

CONVEGNO TORRICELLI IERI E OGGI



EN VIRESCIT GALILÆVS ALTER
Anagn.
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Società Torricelliana di Scienze e Lettere

Questo quaderno raccoglie le presentazioni fatte al Convegno

TORRICELLI IERI E OGGI

svoltosi il 7 e 8 maggio 2025 a cura della Società Torricelliana di Scienze e lettere, della Palestra della Scienza del Comune di Faenza, del Dipartimento di Fisica dell'Università di Bologna, con il patrocinio dell'Associazione ex-Allievi Liceo Torricelli, all'interno della XXV Settimana della Cultura Scientifica e Tecnologica.



Al centro del Convegno la presentazione del Volume “**Homage to Evangelista Torricelli’s Opera Geometrica, 1644 – 2024**” a cura del professor Raffaele Pisano, Università di Lille e del professor Paolo Bussotti, Università di Udine.

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<https://www.torricellianaafaenza.it/w/incontri-culturali-2025/#leslide>

TORRICELLI, chi era costui?

Giovanni Pezzi

Società Torricelliana di Scienze e Lettere

Tappe principali della vita di Torricelli

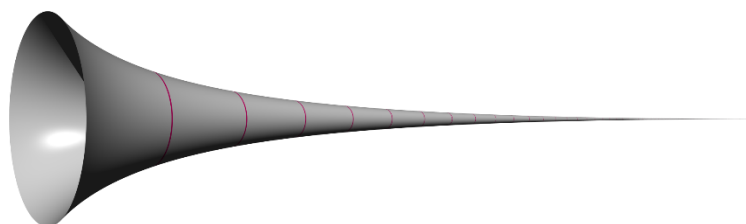
Evangelista Torricelli nacque a Roma il 15 ottobre 1608, come è attestato dal certificato di nascita rinvenuto dal prof. Bertoni (¹). Dopo pochi anni, fu inviato dalla madre, Giacomina Torricella, a Faenza dove fu educato da uno zio della madre, don Jacopo, monaco camaldolese e studiò presso il Collegio dei Gesuiti di Faenza. Nel 1626 si trasferì a Roma, dove rimase fino al 1641. In quel periodo lavorò e studiò con padre Benedetto Castelli e poi fu segretario di Giovanni Ciampoli, che Torricelli seguì nei suoi incarichi governativi nelle Marche e nell'Umbria. Nell'ottobre 1641 si trasferisce a Firenze, presso la residenza di Galileo, di cui diviene assistente per pochi mesi, in quanto Galileo muore l'8 gennaio 1642. Torricelli ne diviene il successore nominato dal granduca Ferdinando II de' Medici matematico del Granducato di Toscana, e poi professore di matematica presso l'Accademia fiorentina. Ma gli "anni mirabili" di Torricelli terminano presto, con la sua morte il 25 ottobre 1647.



I contributi di Torricelli

Nel campo della Matematica il lavoro di Torricelli apparirà negli altri articoli di questo quaderno. Qui, in sintesi, si può citare lo sviluppo del metodo degli indivisibili, i primi passi verso quello che diventerà il calcolo infinitesimale, la quadratura della parabola, della cicloide, lo sviluppo della parabola di sicurezza.

Grande fama viene dalla scoperta della 'tromba di Torricelli', solido iperbolico acutissimo, con volume finito ma area infinita. Questo solido ha la particolarità di avere volume finito, ma area infinita.



Nel campo della Fisica lo ricordiamo per lo studio del moto dei corpi, della velocità di fuoriuscita di un liquido da un contenitore (Teorema di Torricelli), e per l'invenzione del barometro, a cui si deve molta della fama che lo circonda. Torricelli fu anche un abile costruttore di lenti.

Torricelli e Faenza

Nel 1858, in occasione del 250mo anniversario della nascita di Evangelista Torricelli, il Comune di Faenza commissionò allo scultore Alessandro Tomba (Faenza 1825 – Firenze 1864) la "messa in opera" di una statua per celebrare l'insigne scienziato faentino. La statua, realizzata a Firenze, premiata con la Medaglia

¹ Giuseppe Bertoni, "La faentinità di Evangelista Torricelli e il suo vero luogo di nascita". Torricelliana n. 38, 1987.

d'argento all'Expo del 1861, fu trasportata a Faenza, con notevoli difficoltà e inaugurata dove si trova oggi, nella piazza di San Francesco, il 6 ottobre 1864.



La piazza di San Francesco e la statua di Torricelli all'epoca della inaugurazione



La statua di Torricelli oggi

Esposizione universale 1908

Il terzo Centenario della nascita di Torricelli è celebrato a Faenza nel 1908 con importanti manifestazioni artistiche e culturali, visitate dal Re Vittorio Emanuele III e da molti turisti e visitatori italiani.



Pochi anni dopo nasce il Museo Torricelliano, con lo scopo di raccogliere e conservare i cimeli dell'Esposizione, e avviene la pubblicazione delle opere di Torricelli.

Nel 1947 nasce la Società Torricelliana, che nel 1974 si trasferisce nell'attuale sede di Palazzo Laderchi.



Curiosità su Torricelli

A Faenza il nome di Torricelli ricorre spesso: dal nome di una via, all'intitolazione del Liceo, della Riunione cattolica Torricelli, della Società torricelliana di Scienze e Lettere, dell'Osservatorio Meteorologico, del Parco scientifico e tecnologico, del Circolo Numismatico, e anche di una Farmacia e una Autocarrozzeria.

In Italia una decina di scuole, da Nord a Sud, sono intitolate a Torricelli.

A Torricelli, inventore del barometro, è stata dedicata una unità di misura della pressione, il torr. A lui è intitolato l'Asteroido 7437 e un cratere, di 22 km di diametro, sulla Luna.

Nel corso degli anni vari sommergibili della Marina italiana hanno avuto il nome di Torricelli.

Esistono le *Torricelli Mountains*, che si trovano a nord dell'Australia, nella parte nord occidentale della Nuova Guinea Papua.

Torricellia è un genere di piante da fiore appartenente alla famiglia delle *Torricelliaceae*, originario dell'Himalaya orientale, del Nepal, del Tibet, della Cina meridionale e anche del Vietnam. E infine ricordiamo i francobolli commemorativi nel 350° anniversario della nascita di Torricelli, a cura delle Poste italiane e dell'Unione Sovietica.



[Torna all'Indice](#)

Torricelli's *Opera Geometrica* (1644): Methods, Procedures, and the Structure of Mathematical Proof

Raffaele Pisano
IEMN, Lille University–CNRS, France
raffaele.pisano@univ-lille.fr

Paolo Bussotti
DIUM, Udine University, Italy
paolo.bussotti@uniud.it

Abstract. This historical–scientific essay is written in the context of the scholarly celebrations marking the 380th anniversary of the publication of Evangelista Torricelli's *Opera Geometrica* (1644–2024), and forms part of recent international initiatives devoted to a renewed historical and historiographical assessment of Torricelli's scientific work. The article examines Torricelli's life and writings within their scientific context, highlighting the Archimedean heritage underlying his geometry and mechanics. In addition, the internal development of the *Opera Geometrica* is analysed through a combined historical and editorial perspective, addressing both the structure of the work and its complex transmission. On this basis, the paper discusses Torricelli's most significant results (1608–1647), especially in mechanics and geometry, and focuses on his methodological approach to proof. In this context, the diversity of procedures adopted by Torricelli emerges as a key element for assessing the originality and methodological independence of his results. The analysis further addresses two achievements of central mathematical significance: the determination of the volume of the *solido acutissimo* and the geometrical treatment of the logarithmic spiral. In order to clarify the scientific genesis and editorial organisation of the *Opera Geometrica*, an ad hoc reconstruction of its contents is proposed, based on the original chronological sequence of Torricelli's writings. This reconstruction provides a framework for reassessing the place of the *Opera Geometrica* in the development of early modern mathematical physics.

Keywords: Torricelli, Archimedes, Cavalieri, *Opera Geometrica*, Physics-Mathematics Relationship, Principle of Virtual Works, Logarithmic Spiral, Infinitesimal calculus

Outline of the Article

This paper is written in the context of the scholarly celebrations marking the 380th anniversary of the publication of Evangelista Torricelli's *Opera Geometrica* (1644–2024; Pisano, Dhombres, Radelet de Grave and Bussotti 2024), promoted by the *Società Torricelliana di Scienze e Lettere* and associated academic institutions, and articulated through invited scientific meetings held in Faenza (7 May 2025) and Bologna (8 May 2025, University of Bologna, Italy). These occasions provided the immediate framework for a renewed historical and historiographical assessment of Torricelli's work.

The paper is organised as follows. First, Torricelli's life and scientific activity are outlined. We then present the *Opera Geometrica*, discussing its internal structure, historiography, and editorial transmission. Next, we analyse the Archimedean foundations of Torricelli's methods and proofs, with particular attention to geometry, mechanics, and infinitesimal techniques. We subsequently present and discuss three proofs given by Torricelli and compare them with the one provided by Roberval, focusing on Torricelli's ability to employ different mathematical methods, including two distinct procedures for squaring the cycloid, in contrast with Roberval's approach; on this basis, any charge of plagiarism can be definitively excluded. The concluding sections assess the historical and methodological significance of the *Opera Geometrica* within early modern mathematical physics.

1. Evangelista Torricelli

Evangelista Torricelli (Roma, 1608 – Firenze, 1647; hereafter also Evangelista) spanned a fruitful period for science. Thanks to the contributions of several scientists and mathematicians, in the second half of the 30s and in the 40s of the 17th century, mathematical technique was substantially improved. However, Archimedes (c. 287 – c. 212 BC) remained a reference point for Torricelli. In this respect

Torricelli is interesting because he represented a new, modern, and open-minded scientist, who tried to rediscover and extend the results of Greek mathematics. The influence of Archimedes's science (Archimedes 1897) and methodology (Archimedes 1912, 2018) on Torricelli is particularly significant and allows us to understand the inspiration of the great Syracusean, and Greek geometry and science more generally, for modern science (Pisano and Capecchi 2010a).

Torricelli was not a misunderstood genius. His value was fully recognised during his lifetime. From 1642, after Galileo's departure, until his death which occurred prematurely in 1647, probably due to typhus fever, he was the Court Mathematician of the (Gran Duke Ferdinand II) Granduca Ferdinando de' Medici (1549–1609). He was in contact with famous scientists and mathematicians, not only in Italy. Especially in the French mathematical milieu, his results in geometry were known and considered of primary importance. Torricelli had an epistolary exchange with some of the most skilled scholars of his epoch such as Pierre de Fermat (1601/7–1665; Bussotti and Pisano 2020) and Gilles Personne de Roberval (1602–1675). That is to say, he was an external member of that cultural-scientific milieu, whose point of convergence was Marin Mersenne (1588–1648). Torricelli had close relations with Bonaventura Cavalieri (1598–1647), the other great Galileian and geometer and the other main Italian mathematician whose *floruit* dates to the late first half of the 17th century. This notwithstanding, only a part of Torricelli's works were published in his lifetime. This occurred in 1644 when the Florence edition of Torricelli's *Opera geometrica* was published. As a matter of fact, that edition did not include the whole of Torricelli's geometrical production. When Torricelli died, he left several manuscript works and outlines of works. Some of them were published in the late 17th, 18th and 19th centuries, but it was not until the beginning of the 20th century that a complete edition of Torricelli's works was published. However, this does not close the story of their publication because, despite being a fundamental point of reference, this edition has some weaknesses. Another edition was published in 1975. An epistolary exchange among the Galileo's disciples was published (Galluzzi and Torrini 1975) by Paolo Galluzzi and Maurizio Torrini (1942–2019), containing several letters by and to Torricelli (see also: Pisano and Bussotti 2012, 2015b, 2015c; Pisano and Bussotti 2017a, 2017b, 2017c; Pisano, Agassi *et al.* 2017).

In what follows we outline the most significant facts to characterize the adventures of Torricelli's production because they stand as an example of the tortuous way in which the complete knowledge of the results and techniques of scholars can be achieved¹.

1.1 *La Torricella*, and Matematical–Physical Studies

Torricelli was born in Rome on 15 October 1608. His parents came from Faenza. In particular, his mother's family was the owned the farm *La Torricella* near Faenza (Emilia Romagna, Italy). His father, Gaspare Roberti came from Bertinoro – a village between Forlì and Cesena –. Evangelista adopted the name of his mother. His parents sent Evangelista to study in Faenza under the guidance of his maternal uncle Gian Francesco (1567–?) who took the name Jacopo when he entered the monastic order of the Camaldolesi. Between 1623 and 1625 Torricelli was trained in mathematics at the Jesuit college of Faenza (though as an external member). He revealed a particular skill in mathematics. In the second half of 1625 Evangelista returned to Rome and met – probably thanks to the mediation of his uncle – the Benedictine Benedetto Castelli (1578–1643), a famous mathematician who was a pupil of Galileo Galilei (1564–1642). From 1626 Castelli was lecturer in Mathematics at the university of Rome at the *Palazzo della Sapienza* (also "Roma La Sapienza"). For six years Evangelista attended the mathematical academy held by Castelli. Here he met other young scholars interested in mathematics such as Raffaello Magiotti (1597–1656; Berlingozzi 1902) and Antonio Nardi (1598–1648?). In his first letter to Galileo dated 11 September 1632 (Galilei 1890–1909, XIV, pp. 386–387) Torricelli

¹ Cfr. Loria and Belloni (Torricelli 1919, "Introduzione"; Torricelli 1975, "Introduzione" by Lanfranco Belloni). For the history of mathematics and Torricellean biography–*notizie*, see Loria (Loria 1897, 1903, 1909, 1915, 1917, 1919, 1922ab, 1923, 1932, 1938, 1944, 1945, 1947), Podetti (Podetti 1914), Agostini (1930abc, 1931, 1949, 1950, 1951abcde), Bortolotti (1921, 1922, 1923, 1924, 1925ab, 1928abc, 1930, 1931, 1937, 1939, 1947), Loria and Vassura (Torricelli 1919–1944), Vassura (Vassura 1945), Visani (Visani (1952), Prociassi (Prociassi 1952, 1953), Taton (Taton 1963), Belloni (Torricelli 1975), Fabbri (Fabbri 1947–1997). See also: Toscano 2008, 2014; Favino 2009, 2019; Rossini 1959, Bertoni 2003, AA.VV. 2009 and their suggested biographical references, as well.

informed the great scientist that, with Castelli, he had studied many of the important mathematical and scientific subjects: Euclid (Fl. 300 BC), Archimedes (c. 287 – c. 212 BC), Apollonius of Perga (c. 240 – c. 190 BC), Theodosius of Bithynia (c. 169 – c. 100 BC) Ptolemy (c. 100 – 170 AD), Johannes de Sacrobosco (c. 1115– c. 1256), Nicolaus Copernicus, (1473–1543), Tycho Brahe (1546–1601), Christianus Severini Longomontanus (1562–1647) and Johannes Kepler (1571–1630). He wrote that he had also read and profoundly meditated on Galileo's *Dialogue Concerning the Two Chief World Systems* (*Dialogo sopra i due massimi sistemi del mondo*; hereafter *Dialogo*). It is likely that he was also instructed in algebra. Torricelli also frequently saw two Jesuit mathematicians of the Collegio Romano, Austrian Christoph Grienberger (1561–1636) and German Christopher Scheiner (1573–1650). He was particularly interested in their reactions to the reading of Galileo's *Dialogo* (Galilei 1632, 1954) which were positive, though Scheiner complained that Galileo was not respectful towards him. Obviously, he was referring to the controversy regarding the sunspots.

From 1632 to 1641 Torricelli was the secretary of monsignor Giovanni Battista Ciampoli (1589–1643), secretary to Pope Urban VIII (1568–1644). Ciampoli had fallen from grace of Urban VIII because he declared his support of Galileo's *Dialogo* and had favoured the publication of Galileo's work. Consequently, he was given several secondary tasks in different parts of the Papal States, and Torricelli travelled with him. In February 1641 Torricelli came back to Rome and began to frequent the mathematical circle around Castelli, with Magiotti and Nardi. Evangelista had written a treatise on the motion of projectiles. Castelli was so impressed by this work, which completed the theory expounded by Galilei in the third Day of *Discorsi e dimostrazioni matematiche* (1638; Galilei 1890–1909, VIII, pp. 39–362) that he sent it to Galileo. The old scientist invited Torricelli to his residence in Arcetri. Here, besides Galileo, Torricelli also encountered Vincenzo Viviani (1622–1703), who was also a guest of Galileo (Viviani 2001). In that period, Torricelli was offered the possibility of becoming the mathematician of the Grand Duchy of Tuscany (Granduca di Toscana) with a salary of 200 scudi. Thus, he began to frequenting high society. He could dedicate his time to research and became a member of several academies, the most prestigious among them being the famous *Accademia della Crusca*, where Evangelista gave several lectures and talks on diverse arguments, such as the problem of the impact, the origin of gravity and the nature of the winds. From 1641 – at the suggestion of Galileo – Torricelli also began to construct lenses for telescopes². In this field he became a true expert and this work was a source of further earnings. In 1644 his *Opera geometrica* was published in Firenze. With this work, Torricelli became a well-known scientist throughout Europe.

2. The *Opera Geometrica*

Let us now give an overview of Torricelli's major mathematical works mathematical works, focusing, in particular, but not exclusively, on the *Opera Geometrica* (Cfr. Loria's and Belloni's works):

- The Content of *Opera Geometrica*;
- A Historiography of Torricelli's *Opera Geometrica* and *Works*;
- *In quibus Archimedis Doctrina in Opera geometrica*;
- Details from a selection of Torricelli's *Opera Geometrica* proofs on main reasoning as most original contributions to geometry applied to mechanics and infinitesimals.

2.1 The Content

The *Opera Geometrica* written both in Latin and, a small part, in the Italian vernacular, has two main topics: Motion and Geometry. Taking into account our previous research, we divided it *ad hoc* in four main parts. For this reason, below we propose the Content of the *Opera geometrica* (Tab. 1) including the main sections of each part, as they are presented by Torricelli. For example, if a section (e.g., the appendices) belongs to arguments dealt with in the initial chapters, but in fact appears at the end of the *Opera geometrica*, then we chose to respect this editorial choice. This choice also takes into account the original editorial pagination, that is the numbering of pages that restarts at any time; at that

² Cfr. Galluzzi 1976; Miniati, Van Helden, Greco and Molesini 2002; Molesini 2010; Paternoster, Rinzivillo and Schettino 1996; Ronchi 1924, 1968.

time, the term "Book (*Liber*)" meant chapter. Hereafter we will continue to use the term Book to avoid any confusion with respect to the modern significance. The main internal divisions of the *Opera Geometrica* are summarised in Tab. 1.

Tab. 1 A Content of *Opera Geometrica* (393 numbered pages)

Sections	Parts	Title	Main Content
Preliminary Matters	Part 0	<i>Serenissimo Magno Duci Etruria</i> (pp. 3–4); <i>Proemium</i> (pp. 5–10). [N.B.: This section also fits with spherical solids. Thus, it may be also considered as the initial part of the topic below].	Preface; Prologue, Six Polygon species, Specifications on definitions and suppositions.
Chapters on the First Elements of Geometry, Motion and Mechanics	Part 1	<i>De sphaera et solidis sphaearalibus, Liber primus</i> (pp. 11–46); <i>De solidis sphaeralibus, Liber secundus</i> (pp. 47–94).	29 Propositions; 52 Propositions.
	Part 2	<i>De motu graviun naturaliter descendetium, Liber primus</i> (pp. 97–153); <i>De motu proiectorum, Liber secundus</i> (pp. 154–243) ³	58 Propositions; 37 Propositions.
Chapters on Geometry and Infinitesimal	Part 3	<i>De dimensione parabolae solidique hyperboloci, problemata duo</i> (pp. 1–84) ⁴ <i>Ad Serenissima Principem Leopoldum ab Etruria</i> (pp. 3–4); <i>Ad Lectorem Proemium</i> (pp. 5–16). <i>Quadratura parabolae, Problema Primum</i> (pp. 17–84) ⁵ <i>Appendix De dimensione cycloidis</i> (pp. 85–90); <i>Appendix Scholium De cycloidibus aliarum specierum</i> (pp. 91–92).	Preface; Prologue, 6 Suppositions–definitions. 21 Propositions. 3 Theorems; Specifications.
	Part 4	<i>De solido acuto hyperbolico, Problema alterum</i> (pp. 93–112); <i>De solido acuto hyperbolico, Problema Secundum</i> (pp. 113–135) ⁶ ; <i>Appendix De dimensione Cochlea</i> (pp. [136]145–150) ⁷	14 Examples, 1 Scholium, 1 Definition; 29 Corollaries; 1 Definition, 5 Lemmas, 1 Theorem, 1 Scholium

Part 1

De sphaera et solidis sphaeralibus, 2 Books

Torricelli analyses several figures generated by a regular polygon inscribed in or circumscribed to a circle which rotates around one of its axes of symmetry. Some of the results expounded by Torricelli were already known to Archimedes (Capecchi and Pisano 2007abc). The most important method used by Torricelli was exhaustion. Though Archimedes was his explicit source of inspiration, the reasonings proposed by Torricelli are original. The material presented by Torricelli in this work can be dated to the year 1641.

Part 2

³ N.B.: The final part is in Italian vernacular language (pp. 216–243).

⁴ N.B.: The numbering of pages restarts from 1.

⁵ N.B.: The pages number 55 and 56 are repeated. The numbering of pages is broken: pages 79 and 80 are missing. Nevertheless the content is saved.

⁶ N.B.: The numbering of pages restarts from 129. Thus it would be pp. 129–135.

⁷ N.B.: The numbering of pages starts with p. 136 and just after with p. 145, so the numeration continues consequently. The last page is 150, but, in fact, the appendix is only seven pages long.

De motu graviun naturaliter descendetium et proiectorum, 2 Books

Torricelli presents his famous principle according to which two heavy bodies mutually linked cannot move spontaneously unless their centre of gravity descends (Pisano 2017; Pisano and Capecchi 2007abc, 2010a). Torricelli solved several problems whose main source of inspiration were Galileo's works, but he invented new and more rigorous methods. The two books are followed by an appendix entitled *De motu aquarum* where Torricelli formulated the law that given a liquid within a vessel with a hole at the bottom, the liquid falls out from that hole with a velocity proportional to the square root of the distance between the free surface of the liquid and the bottom of the vessel.

Part 3

De dimensione parabolae solidique hyperbolici, 2 Books, 2 Problems, 2 Appendices

This is a very original section. Torricelli used the technique of Cavalieri's indivisibles extending it beyond the rectilinear indivisibles, so creating the curved indivisibles. In fact, Cavalieri in one circumstance introduced curved indivisibles. However, he did not develop a dedicated theory and there are some doubts whether he considered them completely legitimate (Cfr. Vita 1972; Andersen 1985, p. 336). The most important result is the proof of the famous theorem of the "acute hyperbolic solid" which has a finite volume though being generated by the rotation of a hyperbola around its asymptote. This contribution is significant also from a historical-methodological point of view because Torricelli claimed that the indivisibles might have been the heuristic procedure through which Archimedes reached his results. In fact, for the quadrature of the parabolic segment, Torricelli proved (Torricelli 1644, III, 1, *Quadratura parabolae*, p. 17; see also Proposition III, p. 32 *et sugg.*) the Archimedean theorem (Archimedes 1897, Propositions 24, p. 251) by means of two different methods and several procedures (see details below).

Part 4

De solido acuto hyperbolico, 2 Problems, 1 Appendix

Torricelli proved that the area of the cycloid produced in the first rotation is equal to three times the area of the generating circle. This result had been communicated to Cavalieri in 1643 (Torricelli 1919–1944, III, Letter to Cavalieri, February 28th, p. 108).

2.2 A Historiography in Conetxt

Besides his theoretical contributions, the other discovery for which Torricelli is famous concerns the barometric experiment (Cfr. Waard 1936; West 2013). It also dates to Torricelli's *anni mirabiles* 1641–1644. The precursor of the experiment can be referenced in the discussion between Giovanni Battista Baliani (1582–1666) and Galileo on the weight of air. In 1614 Baliani asked Galileo whether he knew a method to measure the specific weight of air. Galileo answered affirmatively in a letter to Baliani on 12 March 1614 (Galileo 1890–1909, XII, p. 35; Torricelli 1975, p. 36). Many years later, in 1630, Baliani told Galileo of the difficulty of raising water with a pump to the height of a hill. Galileo answered that this was the case because the column of water breaks under its own weight. He established 18 *braccia* to be the greatest height to which a column of water could be raised by a siphon. According to Galileo, such a maximal height was an indication of the vacuum resistance. In his opinion, the water's cohesion was due to the attraction produced by the vacuoles interposed between the particles of water. When the weight of the water column overcomes such a cohesion-force, the column breaks. In contrast to this, Baliani was convinced that the impossibility of raising water higher than 18 *braccia*, equivalent to 10.8 meters, depended on the weight of air. He claimed that air weighs on the bodies, which are on the Earth's surface, like water weighs on the bodies, which are on the seabed. Torricelli knew this discussion. He was convinced that Baliani's arguments were correct and tried to determine exactly the pressure exerted by the atmosphere on a point of the Earth's surface in the direction joining the point to the earth's centre. Given the difficulty to form and experiment with a water column of 10.8 meters, Torricelli conceived and developed, jointly with Viviani, his famous experiment with a column of quicksilver (mercury) of the same weight as a column of water 10.8 meters high. Thus, he considered a column of mercury of height 1 and $\frac{1}{4}$ *braccia*, namely 76 centimetres. The experiment did not fully succeed because the vacuum in the tube constructed by Torricelli was not perfect. It was, however, sufficient to prove that all the ideas connected to the *horror vacui* were wrong. Pascal who, on the air's pressure, wrote the two

fundamental memoirs *Traitez de l'Equilibre des Liqueurs et de la pesanteur de la masse de l'air*⁸ and executed the celebrated experiment of the *Puis-de-Dôme* was influenced by Torricelli's work, though he did not mention Torricelli. It is known that two letters to Michelangelo Ricci, in which Torricelli described his ideas and his experiment, were sent by Ricci himself to Mersenne who almost surely informed Pascal.

Torricelli was a versatile scientist, and his opinion was requested with regard to the drainage of the Valdichiana to free that zone from stagnant water. There were two opposite ideas: 1) to drain the river Chiana; 2) to raise, by flood, the level of marshy land to achieve the necessary slope so that water might flow towards the Arno. Alessandro del Borro (1600-1656), who was the chief of the Granduca Ferdinando's army, and Famiano Michellini (1604-1665), lecturer of mathematics at the university of Pisa (he became professor from 1648) were favourable to the first solution. Whereas, Torricelli and Andrea Arrighetti (1594-1672), a disciple of Galileo, who was the intermediary in the correspondence between Galileo himself and his daughter Maria Celeste, thought the second solution to be far preferable. The uncertainties and acrimony of this dispute had the consequence that no decision was taken until 1789 when the solution proposed by Torricelli was adopted.

The relations of Torricelli with the French mathematicians had been very good, but from 1646 the situation changed: Roberval vindicated his priority for the quadrature of the cycloid and accused Torricelli of plagiarism. Roberval and René Descartes (1596–1650) began to doubt that the Galileian law on the free fall of bodies and the parabolic trajectory of projectiles was correct because the experiments did not fit with the theoretical predictions. There was a brief epistolary exchange in which the answers from Torricelli were not decisively in favour of Galileo's law. Then, Torricelli began to write a systematic work to answer all the doubts of the French mathematicians entitled *Racconto d'alcuni problemi proposti e passati tra gli matematici di Francia et il Torricelli ne i quattro anni prossimamente passati* (Torricelli 1919-1944, III, pp. 1-32). However, death, after a brief illness, took Torricelli in the night between 24 and 25 October 1647, so he could not conclude his work.

Torricelli had appointed Ludovico Serenai, who was a close friend, testamentary heir in around 1640. He left precise instructions on the best technique to construct lenses as well as on the publication of his scientific manuscripts. He wanted their publication to be edited by Bonaventura Cavalieri and Michelangelo Ricci (1619–1682). But Cavalieri died shortly after Torricelli, on 30 November 1647. This was when the adventures and misadventures of Torricelli's manuscripts began.

Serenai tried to convince Ricci, through Magiotti as intermediary, and later writing directly to Ricci. In a letter written the 11 April 1648 Ricci communicated to Serenai that he was no longer interested in mathematics and that several family commitments prevented him undertaking difficult task of editing Torricelli's manuscripts. Serenai then proposed the work to Magiotti, but he also declined. Finally, Serenai tried to involve Viviani in the difficult enterprise. After many doubts, Viviani decided that all Torricelli's manuscripts should be copied and sent to him because he did not want to have sole responsibility for their management. Serenai, who was a jurist and did not have any mathematical knowledge, accepted the heavy duty of copying the manuscripts and sending them to Viviani. However, Viviani did not produce the edition. He only published three theorems by Torricelli drawn from the booklet *De proportionibus* within his work *Quinto libro degli Elementi di Euclide ovvero Scienza universale delle Proporzioni spiegata con la dottrina di Galileo* (Viviani 1674; Podetti 1914). Loria developed different hypotheses with regard to Viviani's behaviour (Cfr. Loria in "Introduzione" to Torricelli 1919-1944, pp. XIX-XXIV): perhaps Viviani was engaged in his own research or had to dedicate much time to other tasks, or his almost maniacal precision induced him to write and rewrite the same passages. It must also be taken into account that his health was not good. The fact is that he did not publish Torricelli's works. Serenai understood that the enterprise appeared increasingly difficult. In 1674 he decided that, after his death, the manuscripts were to be given to Agostino Nelli (1661–1725) and, in case of his departure, to Ridolfo Paganelli (1618–1693) and, eventually, to Carlo Roberto Dati (1619–1676). Serenai died on 28 February 1685 and Viviani on 22 September 1703. The subsequent circumstances of Torricelli's manuscripts have some inexplicable aspects: when Agostino Nelli died, his son Giovanni Battista who was a disciple of Viviani, convinced Viviani himself to accept the fiduciary deposit of the Torricelli's manuscripts. This is rather strange, given that Viviani had always refused to keep above cited manuscripts. But, as Loria claimed, what is even more astonishing is that Viviani left precise testamentary indications as to the whole of his

⁸ The *Traitez* (*Traité*s) was published in 1663, one year after Pascal death (Cfr. Pascal 1998-2000).

patrimony, but he never mentioned Torricelli's manuscripts. As a consequence of this serious negligence of Viviani, the manuscripts passed to the abbot Jacopo Panzanini (fl. first half of the 18th century) nephew of Viviani and lecturer of mathematics in Firenze (Florence). In the meantime, in 1715, Tommaso Bonaventuri (1675-1731), a Florentine erudite, member of the *Accademia della Crusca*, who jointly with Benedetto Bresciani (1658-1740), Giuseppe Averani (1662-1738) and Guido Grandi (1671-1742) edited the first Florentine edition of Galileo's works, published Torricelli's *Lezioni accademiche* (Torricelli 1715, 1823; see also Torricelli 1864). When, in 1733 Panzanini died, he left the manuscripts to his nephews Carlo and Angelo. They did not guess the value of that material and sold a part of it, together with some of Galileo's manuscripts, to a grocer. He used a manuscript of Galileo to wrap a piece of mortadella, which was bought by Giovan Battista Clemente Nelli (1725-1793), a Florentine academic, who was a bibliophile and an antiquarian. His most important work was *Vita e commercio letterario di Galileo*, published in Florence in 1793, though, in fear of ecclesiastic censorship, he indicated Lausanne as place of edition. This work is important because the biography of Galileo, based on a deep analysis of his letters, is reconstructed. Nelli, in that piece of paper in which his mortadella was wrapped, recognized Galileo's handwriting. Through the grocer, he contacted Panzanini and acquired the whole material, which included Torricelli's manuscripts. In a letter to Giovanni Lami (1697-1770) dated 15 September 1750 (Nelli 1753, II, p. 763, see also footnote 3) Nelli mentioned all the manuscripts he had bought. Comparing this list with that written by Serenai and Viviani, it is evident that almost all the material was saved. Nelli tried to organize an edition of the Torricelli's manuscripts, but without success.

In 1768 the works of Torricelli on the drainage of the Valdichiana (*Scrittura sopra la Chiana et sgg. his replies on the subject*) were published in the fourth tome of the multi-author work *Raccolta d'autori che trattano del moto delle acque* (AA.VV 1768, IV, pp. 115-118; pp. 125-142; pp. 143-147; pp. 157-166; see also Torricelli 1919-1944, IV). In his testament, Nelli required that his heirs sell the manuscripts to some private citizen only if the Great Duke of Tuscany refused to buy them. In 1818 Ferdinand III Great duke of Habsburg-Lorraine (1769-1824) acquired the manuscripts. They were preserved in the *Biblioteca Palatina*, created by Ferdinando III himself. In 1861 the *Biblioteca Palatina* was merged with the *Biblioteca Magliabechiana*, because of a provision by Francesco De Sanctis (1817-1883), the Minister of Education of the new-born Italian state. In 1864 a monument to Torricelli was inaugurated in Faenza and Giovanni Ghinassi (1809-1870) published a biography of Torricelli with some scientific letters (Ghinassi 1964). In 1875 Baldassarre Boncompagni (1821-1894) published a further selection of the scientific letters by Torricelli⁹. Charles Henry (1859-1926) published another letter by Torricelli. In our context, the work by Raffaello Caverni (1837-1900), *Storia del metodo sperimentale in Italia* (Caverni 1891-1900) is important because the author reported several sections of Torricelli's unpublished works. In 1900 Gino Benedetto Loria (1862-1954) edited Torricelli's studies on the logarithmic spiral (Loria 1900). In 1903 Loria himself planned to institute the National Edition of Torricelli's works and the eighth Section of the *Congresso internazionale di scienze storiche* (*International conference of historical sciences*) held in Roma (Rome), 1903, proposed to the Italian government that the *Accademia dei Lincei* examine Torricelli's manuscripts to decide the best way to prepare the National Edition. The Education Minister accepted to finance this explorative work which was entrusted to Giovanni Vailati (1863-1909) in 1904. However, in 1905 Vailati became a member of the *Commissione per il riordinamento degli studi secondari in Italia* (*Commission for the rearrangement of the secondary studies in Italy*) and the idea of the National Edition of Torricelli's works ran aground. The following year, 1906, as a consequence of a proposal by Giuseppe Vassura (1866-1949) the Conference of the *Italian Physical Society* proposed to the Italian government to prepare an edition of Torricelli's works. Such a proposal had no answer from the Italian government. At this point the Municipality of Faenza, on the occasion of the tercentenary of Torricelli's birth, decided to publish his works. This enterprise was entrusted to Giuseppe Vassura. The work-plan was to publish three tomes, the first one containing the geometrical works, the second the *Lezioni accademiche* (Torricelli 1919-1944, II), the writing on mechanics and other various writings. Finally, the third volume should have contained Torricelli's scientific correspondence.

⁹ The 8th tome of his *Bullettino* Boncompagni published "Intorno ad alcune lettere di Evangelista Torricelli, del P. Marino Mersenne e di Francesco Du Verdus"; "Lettere di Evangelista Torricelli al P. Marino Mersenne"; "Lettere del P. Marino Mersenne ad Evangelista Torricelli"; "Lettere di Francesco Du Verdus ad Evangelista Torricelli." (Boncompagni 1875, pp. 353-456).

Vassura worked for four years on the edition and completed the second and the third tomes. However, in 1912 Vassura had to leave Faenza and other tasks prevented him from continuing his work. To avoid the enterprise failing Vassura suggested that the Faenza Municipality contact Gino Loria to edit the *Opera Geometrica* and the geometrical manuscripts. (It seems that) after many doubts and much indecisions Loria accepted, and in 1919 the *Opere di Evangelista Torricelli* were published in four volumes, 5 tomes by Stabilimento Tipografico G. Montanari in Faenza, edited by Gino Loria and Giuseppe Vassura (Torricelli 1919–1944). In 1944 a fourth tome edited solely by Vassura was added.

1919. Vol. I, Part I. This is subtitled as *Geometria*. It is edited by Gino Loria. *De Sphaera et solidis sphaeralibus libri duo* (Ivi, pp. 1–87); *De dimensione parabolae* (Ivi, pp. 91–162) also including the two appendices on the dimension of the cycloids (Ivi, pp. 163–172); *De solido hyperbolico acuto* (Ivi, pp. 173–221) with the appendix *De dimensione cochleae* (Ivi, pp. 223–230). These works belong to the 1644 *Opera Geometrica*. Whereas the following writings were published for the first time: Appendix to the Lemma XX in *De dimensione parabola* (Ivi, pp. 231–238); *De Tactionibus* (Ivi, pp. 239–290); *De proportionibus liber* (Ivi, pp. 295–327); *De planis varia* (Ivi, pp. 329–345); *De solidis varia* (Ivi, pp. 347–363); *De circulo et adscriptis* (Ivi, pp. 365–375); *De comparatione perimetrorum cylindri, conii ac sphaerae* (Ivi, pp. 377–386); *De aequalitate perimetrorum cylindri, conii ac sphaerae varia*. (Ivi, pp. 387–407);
1919. Vol. I, Part II. This part is also subtitled as *Geometria*. It is also edited by Gino Loria. It only includes previously unpublished works. *Campo di tartufi* (Ivi, pp. 1–43); *Contro gl'infiniti* (Ivi, pp. 45–58); *Sugli isoperimetric* (Ivi, pp. 49–55); *De centro gravitatis sectoris circuli* (Ivi, pp. 59–77); *De maximis et minimis* (Ivi, pp. 79–97); *Nova per armillas stereometria* (Ivi, pp. 107–173); *De centro gravitatus planorum ac solidorum* (Ivi, pp. 175–226); *De infinitis hyperbolis* (Ivi, pp. 227–274); *De infinitis parabolis* (Ivi, pp. 275–328); *De cycloide*; (Ivi, pp. 329–334); *De Hemyhyperbola logaritmica* (Ivi, pp. 335–347); *De infinitis spiralibus* (Ivi, pp. 349–399); *Sezioni coniche* (Ivi, pp. 403–414); *De indivisibilibus* (Ivi, pp. 417–432); *Miscellanea* (Ivi, pp. 433–440); *Miscellanea*. Appendice: *Lettera di Filaleti a Timauro Antiata, Della vera storia della cicloide e della famosissima esperienza dell'argento vivo*¹⁰ (Ivi, pp. 441–482).
1919. Vol. II. It is edited by Giuseppe Vassura. *Lezioni accademiche* (Ivi, pp. 1–100); *De motu gravium naturaliter descendendum et proietorum* (Ivi, pp. 101–102); *De motu gravium naturaliter descendendum* (Ivi, pp. 103–152); *De motu proietorum* (Ivi, pp. 153–232); *De motu ac momentis varia* (Ivi, pp. 233–250, which are also included in the *Opera Geometrica*, 1644); *Soluzione di un problema intorno alle cose che stanno nell'umido* (Ivi, pp. 251–262); *Scritture sulla bonificazione della Val di Chiana* (Ivi, pp. 263–310) as writings on the Valdichiana's draining and a brief, but interesting treatise on practical perspective *Prospettiva pratica* (Ivi, pp. 311–endbook).
1919. Vol. III. It is edited by Giuseppe Vassura. *Racconto d'alcuni problemi proposti e passati scambievolmente tra gli matematici di Francia, et il Torricelli ne i quattro anni prossimamente passati* (Ivi, pp. 1–32); *Carteggio scientifico* (Ivi, pp. 33–510); *Indice cronologico del carteggio* (Ivi, pp. 511–516); *Indice alfabatico del carteggio* (Ivi, pp. 517–endbook)
1944. Vol. IV. It is edited by Giuseppe Vassura in a single volume and consists in three internal editorial parts. *Prefazione al Vol. IV* (Ivi, pp. 9–18); *Documenti alla Vita* (Ivi, Part, I, pp. 19–122); *Documenti alla Opere* (Ivi, Part II, pp. 123–198); *Appendice*, including scientific correspondence (Ivi, Part III, pp. 199–217; among which are *Ricordi dettati da Torricelli a L. Serenai* with the last will of testament of Torricelli and *Notizie suggerite da Vincenzo Viviani a Lodovico Serenai*, which concern Torricelli's life); *Scritture tolte da un codice mss del XVII secolo* (Ivi, pp. 218–230); *Pubblicazioni moderne* (Ivi, pp. 231–338; they are modern essays by Mario Gliozzi, Amedeo Agostini and Ettore Bortolotti); *Pubblicazioni da consultare* (Ivi, pp. 339–endbook).

The Faenza edition is the most complete available and, therefore, it is a fundamental reference point for exploring Torricelli. Nonetheless, it has been subject to several criticisms. In particular, Part II of Vol. I, edited by Loria, which contains Torricelli's geometrical manuscripts was criticized. Before his death, Torricelli did not have time to put his manuscripts into rational order, and Loria transcribed them with a very limited number of interventions. Ettore Bortolotti (1866–1947) for one, claimed that the very few interventions by Loria are misleading and that his annotations are wrong. He wrote explicitly:

¹⁰ Cfr. Torricelli [1644]1897.

The manuscripts remained available to the scholar in the National Library of Florence for a long time. After more than two centuries, they have been published in the Faenza edition of *Opere Geometriche*. However, the typical disorder of the unpublished writings was made worse by the editor's work, who did not penetrate the Author's thought. This notwithstanding, he wanted to intervene and omitted necessary things, while repeating several times (without realizing it) things of low or no value. Furthermore, his annotations are a mere phantasy. In contrast to this, the pure reproduction of the manuscript would have been a meritorious work.¹¹

As Lanfranco Belloni (1944–2017) points out (Torricelli 1975, pp. 20–21), there were two aspects to Bortolotti's judgement: first of all, Bortolotti and Loria had two different opinions of Torricelli's geometrical works. Bortolotti thought that many elements of integral calculus were already present in Torricelli; whereas Loria considered Torricelli an ingenious mathematician, but regarded his methods closer to those of Archimedes than the inventor of infinitesimal calculus. Furthermore, Bortolotti accused Loria of having carried out his work of editor without any enthusiasm for penetrating the real meaning of Torricelli's methods. The Faenza edition had a troubled history, and it is true that Loria could have presented Torricelli's manuscripts with more care, in order to provide a rational order to Torricelli's works. However, the edition, such as it is, for the moment, is an irreplaceable document. In 1955, Ettore Carruccio (1908–1980) offered an improved edition in comparison to Loria's *De infinitis spiralibus* (Torricelli 1955). More recently, the Italian translation of the *Opera Geometrica*, edited by Belloni was published in 1975. This precious edition also contains an Italian translation of other geometrical writings by Torricelli. The most important concern indivisibles and the logarithmic spiral. Furthermore, the academic lectures, the letters on vacuum and the writings on the draining of the Valdichiana are also added. The historical notes by Belloni as well as his Introduction are documents that enrich this edition and make it a significant reference point for Torricelli research.

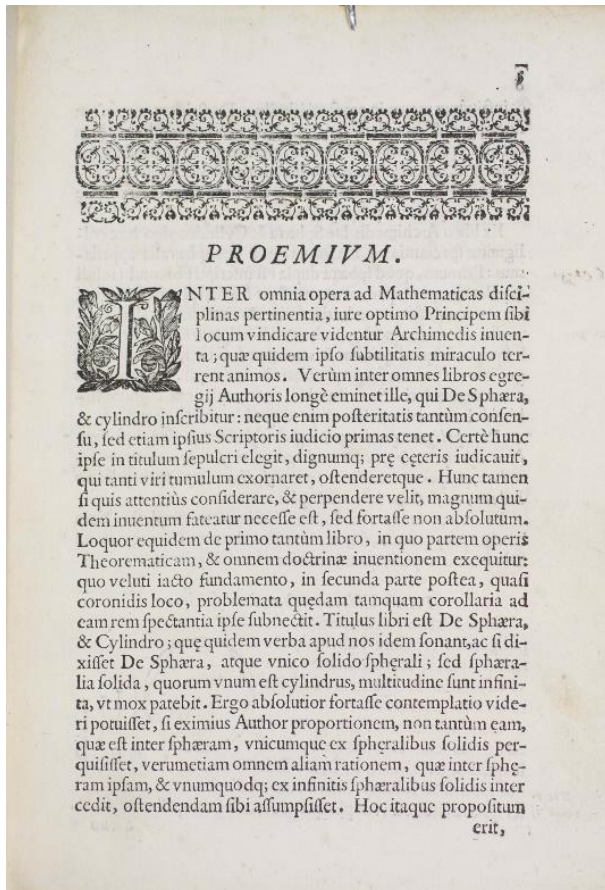
The Loria–Vassura edition of Torricelli's *Opere* does not offer a translation into a modern language. Torricelli's works are presented in their original Latin and vernacular Italian language: a crucial reference for historians of mathematics and physics. Our present Torricelli Springer Project¹² at this stage, includes a selection of outstanding papers on the subject and a commented transcription of the *Opera geometrica* only; all in a unique international volume. For the future, in the long term and if possible, we are thinking—and—scheduling the English commented translation of both the *Opera geometrica* and the whole of Torricelli's *Opere*; this is possible because distinguished scholars such as Loria, Vassura, Bortolotti, Agostini *et al.*, before us, analysed and archived mountains of documents. We hope to do that for our posteriors, as well.

2.3. *In quibus Archimedis Doctrina*

In this section we discuss the role played by Archimedes's mechanics in Torricelli's works (Pisano and Capecchi 2010a, Capecchi and Pisano 2007abc; et al. Pisano's works below). It was truly a scientific inspiration for Torricelli as he himself declared from the beginning of the *Opera Geometrica* (Fig. a)

¹¹ “I manoscritti di Torricelli sono rimasti a lungo nella Biblioteca Nazionale di Firenze, a disposizione degli studiosi; dopo più di due secoli hanno visto la luce, nella edizione faentina delle *Opere Geometriche*. Ma il caratteristico disordine degli scritti inediti fu aggravato dall'opera dell'editore, che non seppe penetrare nel pensiero dell'Autore, e pur volle metterci mano, omettendo cose necessarie, ripetendo a più riprese (senza nemmeno accorgersene) cose di scarso o nessun valore, e postillando fantasticamente. Mentre sarebbe stata opera meritoria, anche la semplice e pura genuina riproduzione del manoscritto, tal quale si trovava.” (Bortolotti 1939, p. 459; Authors' translation).

¹² In 2009 the Torricelli-Project was born thanks to Raffaele Pisano's personal initiative. In Florence, he met the outstanding Galilean and Torricellian scholar, Prof. Maurizio Torrini (1942–2019; Museo Galileo, Florence, Italy) who welcomed the Torricelli-Project with profound interest. A correspondence (July 2010) exists, as well. In the future, the Torricelli Project also envisages a second and third part respectively, the commented translation, side-by-side into English of the *Opera Geometrica* and the commented translation, side-by-side of the whole *Opere di Evangelista Torricelli* (Torricelli's Works).



"Among all the works pertaining to the Mathematical disciplines, those of Archimedes seem to claim the first place; indeed, with their miraculous subtlety they throw in dismay the minds."

Fig. a. Torricelli Inspiring by Archimedes's Science

(Torricelli 1644, *Proemium*, p. 7; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 302-303; Authors' translation)

Source: Courtesy University of Oklahoma, History of Science Collections

Below the reader will find in the detail the influence exerted by Archimedes through the centuries and his heritage in Torricelli's works, including his approach to solving problems and conducting proofs.

From the history of physics—and—mathematics standpoint, the late 30s and the 40s of the 17th century represent a particularly rich period of research for the development of science. Mathematical techniques advanced rapidly, compared to the beginning of the century. Nevertheless, infinitesimal calculus (Bussotti 2019) was not yet available. A few years later this branch of mathematics will undergo a spectacular development which will allow scientists to solve specific problems (e.g., the calculations of areas, volumes, centre of gravity) and, very importantly, to deal with instantaneous physical–mathematical quantities (i.e., acceleration). Scholars understood that *finitary techniques* were not enough to solve certain problems addressed by the new science and the new mathematics. By *finitary technique*, we mean a technique through which it is possible, e.g., to prove the equivalence of two planes of solid figures through the *equidecomposition* in a finite number of parts. Beyond this, they referred to the historically available technical model: the Archimedean. In this context, the *exhaustion method* was particularly significant for two main reasons:

- a) Many problems were solved by this method, as we will see below with regard to Torricelli;
- b) The exhaustion method and its potentially infinite procedures were a source of inspiration for techniques, which were slightly different from the exhaustion method itself.

For, Cavalieri, earliest credited inventor of the indivisibles method, expressed to Galilei in 1621, his interest in the field:

[...] I am proving some Archimedes' propositions, in a different manner with respect to him, and in particular those on the quadrature [of the segment of the] parabola, [and] also different from those of Y.L.; [...].¹³

Archimedes had also been the inspiration of Kepler and Galileo themselves (Bussotti 2015; Pisano and Bussotti 2017a). For example the infinitary techniques used by Kepler, especially in the *Nova Stereometria doliorum* (1615), has the Syracusan scientist as a reference point (Pisano and Bussotti 2012, 2017c). When calculus became the standard paradigm of the infinitary procedures (which happened around the 80s of the 17th century), references to Archimedes became fewer and less direct. Torricelli's works (around 40s) represent one of the most accomplished results achieved without resorting to calculus.

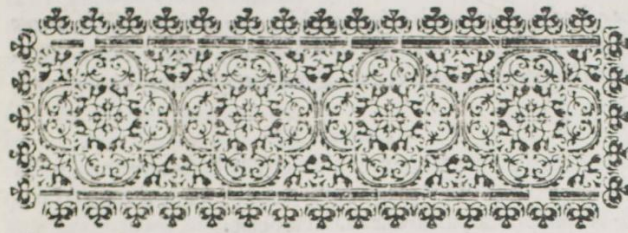
From the point of view of historical epistemology, in some aspects Archimedes breaks the geometrical paradigm with respect to Euclid. If we interpret *normal science à la* Kuhn as Euclidean geometry, namely a discipline based upon a system of axioms, which rely on self-evidence, then Archimedes offers a different scientific paradigm. The birth of new modern science, whose necessity was not only to develop pure mathematics, but also to apply mathematics to the external world and to technique, adopted a paradigm *à la Archimedes* rather than *à la Euclid*. This means that the basic principles did not derive from self-evidence, only, but they were also drawn from experience. This is the case with Torricelli (see below), but it also happens within Tartaglia's *Science of weights*, i.e., Books VII-VIII of the *Quesiti et invenzione diverse* (Tartaglia [1554]1959; Pisano 2020, 2016; Pisano and Capecchi 2016) and in Galileo's mechanics, as well (Pisano and Bussotti 2017a, 2015b, 2012).

The concepts themselves of mathematical proof and of mathematical rigour were almost lost until the 16th century and they were reacquired, with difficulty, through a process which – with regard to mathematical disciplines other than calculus – stretched into the 17th century with the fundamental contributions of Fermat (Bussotti 2006; Bussotti and Pisano 2020) and Descartes (Bussotti and Pisano 2013; Schuster 2013ab, 2000). In this phase, the Greek scholars represented a model of rigour and creativity, which was an important guide for the scientists living in the 16th and 17th century. As to calculus and more modern branches of mathematics, the rigourization-process was completed only in the 19th century (Pisano 2015, 2016, 2017; see also Pisano Coopersmith and Peake 2021).

Archimedes was fundamental for Torricelli's works, and for this reason his influence on the Italian scientist is so significant from a historical-scientific standpoint, particularly with regard to the mathematics of indivisibles. Indeed, Torricelli's approach was to confront geometric matter both with the methods of the ancients, in particular the exhaustion method, and with indivisibles, so attempting to compare the two, as is clearly seen in his letters to Cavalieri (Torricelli 1919-1944, III). For example, we also note his pedagogical attitude to solving problems by different methods (Figs. bcdef). He solved in twenty-one different ways Archimedes's Proposition 24 squaring the parabolic segment¹⁴: 11 proofs by the method of exhaustion, 10 proofs by Cavalieri's method of the indivisibles (Cavalieri 1635, IV, Theorem I, Proposition I, p. 285):

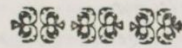
¹³ "[...] vado dimostrando alcune proposizioni d'Archimede diversamente da lui, et in particolare la quadratura della parabola, divers'ancora da quello di V.S; [...]" (Letter from Cavalieri to Galilei, 1621, 15 December. Galilei 1890-1909, XIII, p. 81; Authors' translation).

¹⁴ Torricelli 1644, III, 1, *Quadratura parabolae pluris modis per duplicem positionem more antiquorum absoluta*, p. 17; *Lemma VII*, p. 28, *Proposition III*, p. 32 *et sugg.*; Archimedes 1897, Propositions (17, p. 246 and) 24, p. 251; see also Proposition 10, p. 162; *Id.* 1912, the *Method*, Propositions I-II, pp. 15-18). For Torricelli's quotation to Valerio see: Valerio 1606. On the subject, see also: Galilei 1890–1909, II, pp. 140–146, pp. 286–294; Vol. VIII, pp. 176–183; p. 306 *et sgg.* (Cfr. Bussotti 2019; Pisano and Capecchi 2010a).



QVADRATVRA P A R A B O L A E.

Pluribus modis per duplicem positio-
nem, more antiquorum, absoluta.



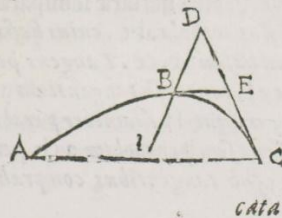
Lemma Primum.



I P A R A B O L A duas tangentes habuerit, alteram ex termino basis, alteram verò ex vertice: tangens, quæ ad basim est, bifariam secabitur ab illa, quæ per verticem ducitur.

Esio parabola abc , cuius diameter bi , ordinatim verò applicata (sive basis) sit ac ; tangens ex termino basis sit cd ; per verticem verò tangens be . Dico ipsam cd bifariam secari in puncto e .

Cum $n.c.d$ sit tangens, di diameter, erunt æquales inter se db . bi . Et quia ac ordinatim appli-



35 primi
Conic.

Fig. b. Torricelli's *Quadratura parabolæ pluris modis* – Starting with Method of the Ancient Geometers (Torricelli 1644, III, p. 17; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 814-815)

Source: Courtesy University of Oklahoma, History of Science Collections.

[The algebraical equivalent of this result is of course

$$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots + \left(\frac{1}{4}\right)^{n-1} = \frac{1}{3} - \frac{1}{3} \left(\frac{1}{4}\right)^{n-1}$$

$$= \frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}.]$$

Proposition 24.

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

Suppose $K = \frac{4}{3} \Delta PQq$.

where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to K .

For, if the segment be not equal to K , it must either be greater or less.

I. Suppose the area of the segment greater than K .

If then we inscribe in the segments cut off by PQ , Pq triangles which have the same base and equal height, i.e. triangles with the same vertices R, r as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K .

Therefore the polygon so formed must be greater than the area K ; which is impossible, since [Prop. 23]

$$A + B + C + \dots + Z < \frac{4}{3}A,$$

where

$$A = \Delta PQq.$$

Thus the area of the segment cannot be greater than K .

II. Suppose, if possible, that the area of the segment is less than K .

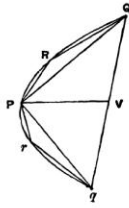


Fig. c. Archimedes *Proposition 24* on Quadrature of the Parabola

(Archimedes 1897, *Proposition 24*, p. 251)

Source: Google Books. Public domain

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illa sibi inscripta; totum sua parte. Quod est impossibile. Aequi-ponderant ergo parabola inuerfa (de mpto) emtriangulo inscrip-
 to) ex puncto quod dictum est, Quod erat ostendend. &c.

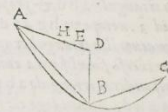
Corollarium

Hinc inferre possumus, quod si ex puncto c , recta ducatur diametro aequidistans, centrum praedictae figurae erit in produ-
 cta. Siquidem figura ex puncto c aequiponderat, & linea ex c ducta aequidistans diametro, est ad horizontem perpendicularis. Posset etiam demonstrari, nisi extra rem esset, centrum praedictae figurae dictam parallelam ita secare, ut pars quae terminatur ad centrum sit ad reliquam ut 11. ad 12.

Propositio III.

Parabola sesquialtera est trianguli eandem sibi basim, & ean-
 dem altitudinem habentis.

Esto parabola ABC , ex qua demptum sit dimidium trianguli inscripti: Sumptaq; DH , quae sit tertia pars totius DA & DE quin-
 ta pars eiusdem; si parabola huiusmodi statuatur inuerfa, ita ut diameter sit hori-
 zonti perpendicularis, aequiponderabit figura ex puncto E . Sed triangulum ABD appensum est secundum centrum grauitatis ad punctum H librae HD . Duae autem paraboli-
 cae portiones residuae appensae sunt secundum centrum grauitatis ad punctum D . Ergo triangulum ABD , ad duas reli-
 quas portiones erit ut DE ad EH , reciprocè, nempe ut 3. ad 2: Sumptis autem antecedentiū duplis erit totum inscriptum trian-
 gulum ad reliquas portiones ut 6. ad 2. Conuertendo igitur, & componendo, erit ipsa parabola ad inscriptum sibi triangulum ut 8. ad 6. Nempe sesquialtera. Quod &c.



grac-
 Lem.

Libet

Fig. d. Torricelli's *Proposition III* on Quadrature of the Parabola

(Torricelli 1644, III, pp. 32-33; Cf. Pisano,

Dhombres, Radelet de Grave and Bussotti, 2024, pp. 844-845)

Source: Courtesy University of Oklahoma, History of Science Collections.

Archimedes's influence on Torricelli can also be detected in two general approaches of both scientists because Torricelli regarded axioms in a similar way to Archimedes's and very differently from the Euclidean approach. Axioms have their origin not only in evidence, but also in experience. Therefore, they are different from, e.g., the epistemically rigid structure of the Euclidean axioms. Additionally, they could also be *local axioms*, namely unproved propositions valid in a given theory, whose validity could be non-universal, again in contrast to Euclid's axioms (Euclid 2007), which were, at that time, believed universally valid. Torricelli exploited Archimedes's techniques in an original manner. Therefore, knowledge of Archimedes's contribution is fundamental to the analysis of Torricelli's *Opera Geometrica*.

Archimedes was (perhaps) the first scientist to set *rational criteria* for determining centres of gravity of bodies and his work contains physical concepts formalised on a mathematical basis (Capecchi and Pisano 2011, 2010b). In *Book I* of *On the Equilibrium of Planes* (Archimedes 1897, I, pp. 189–202) he, besides studying the rule governing the law of the lever, also finds the centres of gravity of various geometrical plane figures (Pisano and Capecchi 2010a; Pisano 2009a). Archimedes's typical method of arguing in mechanics was by the use of reduction *ad absurdum*. Torricelli in his study on the centres of gravity (Torricelli 1644, II, 1, pp. 97-153, II, 2, pp. 154–243) also adopted this *ancient* Archimedean approach. Particularly, in *De motu grauium naturaliter descendetium et proietorum, Liber primus* (*Ibidem*) Torricelli's famous premise is expounded (Torricelli 1644, II, 1, p. 99) and the early foundations of the *Principle of virtual laws* were modelled (Pisano 2017; Capecchi and Pisano 2010a, 2007b).

In *Archimedis Quadratura parabolae* (Archimedes 1897, pp. 233–252) Archimedes also first obtained his results using a mechanical approach and then reconsidered the arguments faced by means of a geometrical method in order to confirm the correctness of his results in a rigorous way. Similarly, Torricelli, with the compelling idea of duplicating the procedure (see above) devoted many pages to prove certain theorems on the *parabolic segment*, by following the geometry used in antiquity (Torricelli 1644, pp. 17-54) and then also proving the validity of his proofs with the *indivisibilium* method (Torricelli 1644, IV, pp. 93-135: Fig. e).



Fig. e. Torricelli's *Quadratura parabolae* by Method of the indivisibles
 (Torricelli 1644, III, p. 55 [N.B.: pages number 55 and 56 are repeated]; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 894-895)
 Source: Courtesy University of Oklahoma, History of Science Collections.

In this respect, it is also interesting to note that Torricelli explicitly and repeatedly underlined Archimedes's methods—and—reasonings (Cfr. Archimedes 1897, *Quadratura parabolae*, pp. 253-252; pp. 55-84; Torricelli 1644, III, *Quadratura parabolae per novam indiuisibilium Geometriam pluribus modis absoluta*, p. 55). For example:

Concordance of the previous proof with the doctrine of Archimedes.¹⁵

Therefore, his main ideas were both to give to the reader results and methods as well as to claim that the indivisibles technique, in a certain manner, was not completely unknown to ancient Greek scholars. Heath, with Heiberg the main scholar and editor of Archimedes, wrote:

[...] the historian of mathematics, in dealing with Archimedes obligations to his predecessors, has comparatively easy task before him But it is necessary, first, to give some description of the use which Archimedes made of the general methods which had found acceptance with the earlier geometers, and secondly, to refer to some particular results which he mentions as having been previously discovered and as lying at the root of his own investigations or which he tacitly assumes as known. (Archimedes 1897, III, p. XI).

Torricelli seems to hold onto the idea that the method of demonstration of the ancients, such as Archimedes' method, was intentionally kept secret. He stated that the ancient geometers worked according to a method "[...] in invenzione [...]" suitable "[...] ad occultandum artis arcanum [...]" (Torricelli, 1644, III, p. 55). However, Archimedean influence in Torricelli goes further. *De sphaera et solidis sphaeralibus* (Torricelli 1644, I, 1, *Liber primus*, pp. 11-46) presents an enlargement of the Archimedean proofs in *On the sphere and cylinder* (Archimedes 1897, II, pp. 1-90). For, we read Torricelli's frontispiece on Archimedes' scientific heritage:

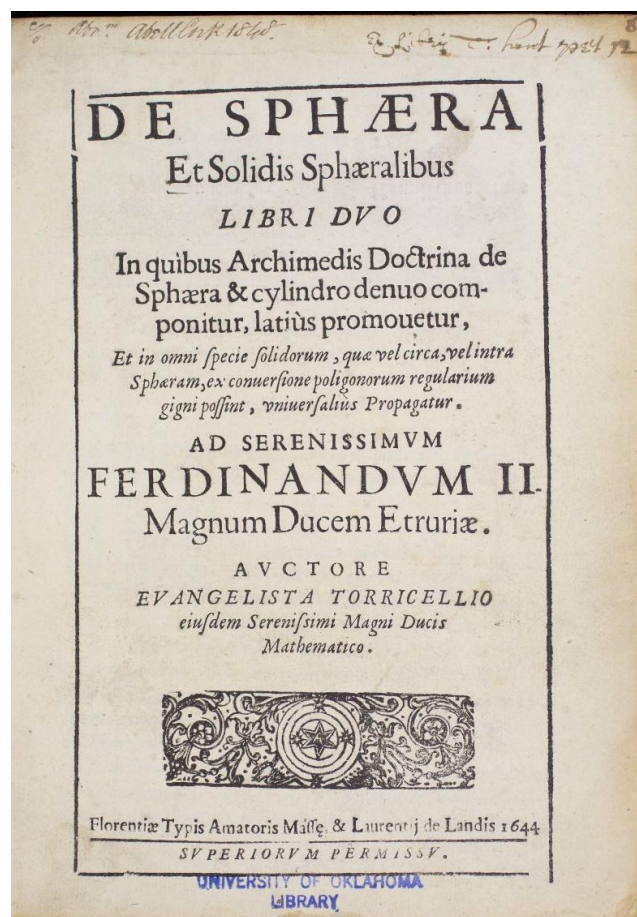


Fig. f. Torricelli Claims his Inspiring by Archimedes's science: *In quibus Archimedis Doctrina* (Torricelli 1644, Frontispiece; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 294-295). Source: Courtesy University of Oklahoma, History of Science Collections

¹⁵ "Concordantia dantia præcedentis demonstrationis cum doctrina Archimedis." (Torricelli 1644, IV, *De solido acuto hypebolico, Problema alterum*, p. 103; Authors' translation. See also pp. 95–102; pp. 108–112). In this Book, see also his claiming on Euclid's proof (*Ivi*, Example IV, p. 97, Example XI, p. 105).

In other domains, Torricelli also faced problems not yet solved by Archimedes, or by the other mathematicians of antiquity. But by adopting the same proofing style as Archimedes, he avoided to formulating the first principles of a theory (i.e., *à la* Euclid). In his words:

Scholium. [...]. The truth of the preceding Theorem [*De solido hyperbolico acuto problema secundum*] is sufficiently clear in itself, and also confirmed by the examples proposed at the beginning of the book. But I think, in order that I may in this part satisfy the reader, who is not a friend of indivisibles [method], I will repeat this very demonstration at the end of the opera [*Opera geometrica*], by the usual way of demonstrating by ancient Geometers.¹⁶

We also remark that mechanical and geometrical argumentations presented by Archimedes in his *Method* (Archimedes 1912) were obviously unknown in Torricelli's time. Johan Ludvig Heiberg (1854–1928) only discovered it in 1906. Therefore, in Archimedes's writing there were lines of reasoning which – because of a lack of justification – were labelled as mysterious by most scholars. Thus, in such instances it was necessary to assure the reader of the validity of the thesis and also to offer a more heuristic approach, given the counterintuitive nature of exhaustion reasoning and *reductio ad absurdum*, by proving Archimedes's results with some other technique. It is well known that Archimedes in the *Method* (Archimedes 1912) faced a series of problems whose solution he anticipated by means of crucial propositions, i.e., based on the determination of centre of gravity and on the equilibrium of the lever. Then he proved them by *reductio ad absurdum*, using, in many cases the exhaustion method (Pisano and Capecchi 2010a). Indeed, Archimedes's himself did not attribute the same amount of certainty to his *Method* of proof as he attributed to other geometrical proofs; his reasoning on *Quadratura parabolae* (Archimedes 1897, Proposition 24, p. 251) is a typical example. At the beginning of the *Method*, addressing Eratosthenes (276–196 B.C.), Archimedes wrote:

Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. (Archimedes 1912, p. 13):

When it comes to Torricelli as an Archimedean scholar, one of his methodologically characteristic proofs was the return to the demonstrative approach followed by the ancient Greeks, with the explicit description of the technique of reasoning actually used. But, what about his mathematical techniques? Besides the well-known *ad absurdum* proof he also used proportions with the classical techniques of the *permutando* and the *ex aequo*. In *De proportionibus liber*, Torricelli defined them explicitly (Torricelli 1919–1944, p. 314).

Generally speaking, Torricelli – like Galilei and Kepler (Pisano and Bussotti 2017a, 2012; Bussotti 2015) – was not interested in algebra and adhered to the language of proportions. In the *De Proportionibus liber* (Torricelli 1919–1944, pp. 295–327) he stressed mathematical language, using only the theory of proportions in geometry. In such a way, he naturally avoided the use of the *plus* or *minus*, whose logical status was still uncertain. Thus, he replaced the *plus* and *minus* by *dividing* (Torricelli 1919–1944, p. 313) and the *composing* (*Ivi*, p. 316). This choice allowed him to work always resorting to the ratio of segments. By following the ancients to sum up segments, he could have thought them as aligned and then translated and connected, making use of terms like “simul”, “et” or “cum” (Torricelli 1919–1944, Proposition, XV, p. 318). We remark that the proofs by means of indivisibles are not *reductio ad absurdum*. Instead, in nearly all other proofs Torricelli used the

¹⁶ "Scholium. [...]. Veritatem præcedentis Theorematis [*De solido hyperbolico acuto problema secundum*] satis per se clarum, & et per exempla ad initium libelli proposita confirmatam satis superq; puto Tamen ut in hac parte satisfaciam lectori etiam Indivisibilium parùm amico, iterabo hanc ipsam demonstrationem in calce operis, per solitam veterum Geometrarum viam demonstrandi longiorem quidem sed non ideo mihi certiorē." (Torricelli 1644, IV, 2, p. 116; Authors' translation).

technique typical of proportions, *dividendo*, *permutando* and *ex aequo*. The correct use of proportions was an important step towards the birth of modern science, in particular in its initial phases. Below we present Torricelli's proofs (Tab. 2) divided by methods and techniques (adapted from Pisano 2011):

Tab. 2 The Main Archimedean proofs by Torricelli in his *Opera Geometrica*

Items	Kind of proof	<i>Opera Geometrica</i> (Torricelli 1644)
Lemma II,V,VI, X-XI, XII-XIII XVII, Propositio IV Lemma XIV	<i>Ad absurdum</i> proof <i>Ex aequo</i> , <i>dividendo</i> and <i>permutando</i>	QUADRATURA PARABOLÆ PLURIBS MODIS [...] ANTIQUORUM absoluta <i>Ivi</i> , pp. 18-54 (<i>Ibidem</i>)
Lemma XVI, XVIII Lemma XIX	<i>Ex aequo</i> <i>Ex aequo</i> and <i>Ad absurdum</i>	(<i>Ibidem</i>) (<i>Ibidem</i>)
Propositio III ¹⁷ Propositio V	<i>Componendo</i> <i>Ad absurdum</i> proof and <i>Componendo</i>	(<i>Ibidem</i>) (<i>Ibidem</i>)
Propositio IX ¹⁸	<i>Ex aequo</i> et <i>Ad absurdum</i>	(<i>Ibidem</i>)
Propositio XVIII Propositio XIX ¹⁹ Propositio XXII-XXIII ²⁰ Propositio IV	<i>Ad absurdum</i> <i>Ad absurdum</i> <i>Ex aequo</i> The same proof as Archimedes ²¹	DE SPHAERA ET SOLIDIS SPHAERICALIBUS <i>Ivi</i> , I, pp. 28-29 <i>Ivi</i> , I, pp. 30-33 <i>Ivi</i> , I, pp. 35-36 <i>Ivi</i> , II, pp. 51-52
Lemma XXII,XXIX ²³	<i>Ex aequo</i>	QUADRATURA PARABOLÆ PER NOVAM INDIVISIBILIIUM GEOMETRIAM PLURIBUS MODIS ABSOLUTA ²² <i>Ivi</i> , pp. 61-72
Lemma XXX Lemma XXXI	Proof based on thoses done by Archimedes, <i>Equiponderants</i>	<i>Ivi</i> , pp. 74-77

¹⁷ In Proposition III, Torricelli quotes Luca Valerio's *De centris gravitatis solidorum libri tres* (Valerio 1604, II, p. 12). He wrote: "Libet hic demonstrare Lemma Lucae Valerij, nostro tamen modo, diversisque penitus Mechanicae principijs. Ipse enim utitur propositione illa, quā ante demonstraverat centrum gravitatis hemisphereij. Nos autem simili ratione, ac in praecedentibus, demonstrabimus et Lemma, et ipsam Valerij conclusionem." (Torricelli 1644, *Quadratura parabolae*, p. 33).

¹⁸ *Lemma* e *Propositio* are proved by means of *Componendo* and *Conversione*.

¹⁹ "Quod est absurdum e contra principia Archimedis." (Torricelli 1644, *De sphaera et solidis sphaearalibus*, I, p. 31).

²⁰ "Tria haec Theoremata, quae sequuntur, ex Archimede desumpta sunt; quod quidem fecimus ne lector Archimedes adire cogeretur, sed universam hanc doctrinam in hoc libello haberet." (Torricelli 1644, *De sphaera et solidis sphaearalibus*, I, p. 35).

²¹ Archimedes 1897, *On the sphere and cylinder*, II, Proposition 25, p. 32. A difference can be remarked: Archimedes proved the case-study, 4-multipolygons, while Torricelli proved the case-study 2-multipolygons.

²² "Hactenus de dimensione parabolae more antiquorum dictum sit; Reliquum est eandem parabolae mensuram nova quedam, sed mirabili ratione aggrediamur; ope scilicet Geometriae Indivisibilium, et hoc diversis modis [...]" (Torricelli 1644, *Quadratura parabolae*, p. 55).

²³ For the other Lemmas/Propositions Torricelli adopted a geometrical construction by means of *componendo* and *dividendo*.

		DE SOLIDO ACUTO HYPERBOLICO PROBLEMA ALTERUM <i>Ivi</i> , I, pp. 93-112
Exemplum I-II, IV-V, X, XII-XIV Other <i>Lemmas</i> and <i>corollaries</i> ²⁴ Exemplum III, XI	Proof <i>in concordantia præcedentis demonstrationis cum doctrina Archimedis Ex æquo</i> (Exemplum X) Dimostrazione <i>in concordantia cum theoremate Euclidis</i>	<i>Ivi</i> , I, 95-108; I, pp. 113-135 <i>Ivi</i> , I, p. 97; I, pp. 104-105
Lemma (From Prop. XXXIII) ²⁶	Proof of Archimedes's proposition on <i>Conoids and spheroids</i> ²⁷	DE MOTU PROIECTORUM ²⁵ <i>Ivi</i> , II, p. 187
Propositio ²⁹ III Propositio VIII Propositio IX Propositio X Propositio XI Propositio XII Propositio XIII-XIV Propositio XV Propositio XVI	<i>Ad absurdum</i> <i>Dividendo</i> <i>Permutando</i> <i>Permutando and Ex æquo</i> <i>Ex æquo</i> <i>Componendo and Conversion</i> <i>Permutando</i> <i>Permutando and Componendo</i> <i>Permutando and Ad absurdum</i>	DE PROPORTIONIBUS LIBER ²⁸ Torricelli 1919-1944. I, p. 308; pp. 313-317).

This table clarifies the chronological organisation of the propositions discussed in the following sections.

The complex historical process called *rediscovery of the ancients* involves many aspects; one of them is the rediscovery of classical Greek geometry with its heritage of results and methods. Actually, the development of the methods and techniques of the ancient geometers were so important in the 17th century that they remained a conspicuous part of the mathematical apparatus used by the physicists until Newton (Pisano and Bussotti 2017b, 2016, Bussotti and Pisano 2014; Pisano 2026, pre-print).

3. The Proofs: A Selection

Below we present a selection from *Opera geometrica* of the proofs and reasoning focusing on the original and significant contributions to geometry with respect to their application to mechanics and infinitesimals.

3.1 *De Sphaera et Solidis Sphaeralibus*

In *De sphaera et solidis sphaeralibus*, the first work of the 1644 edition, Torricelli, extending the method of exhaustion (in *Archimedis Doctrina*, as above commented) and proved a series of theorems on the solids obtained by the rotation of a regular polygon inscribed in or circumscribed to a circle. Thus, solids of rotation inscribed in or circumscribed to a sphere are studied. This programme is clearly expounded by Torricelli in the *Proemium* of his work.

²⁴ The others are not *ad absurdum* proofs. They are proved by geometrical construction.

²⁵ Torricelli 1644, *De motu proietorum*, II, pp. 154-190.

²⁶ “Propositio Archimedis est sequens lemma in libro de Sphaeroidibus et Conoidibus, quam tamen expeditius demonstrabimus.” (Torricelli 1644, *De motu proietorum*, II, p. 187).

²⁷ Archimedes 1897, pp. 99-150.

²⁸ In the end of this book, Torricelli claimed for the readers the *concordantia* with Euclidean ones. (e.g., propositions XVII and XVIII, XXIV and XXV of *Elements*, Book V).

²⁹ The others are not *ad absurdum* proofs. They are proved by geometrical construction.

Torricelli individuates six species of revolution solids: those of the first and second species are given by the rotation respectively along the apothem (*Catetum*, according to Torricelli) and the diagonal of an even-sided regular polygon inscribed in a circle. That of third species is given by the rotation of an odd-sided regular polygon inscribed in a circle and rotating around the apothem. Those of the fourth and fifth species are analogous to the ones of the first and second species, but they are circumscribed. That of the sixth species is analogous to the solid of the third species, but it is circumscribed. Torricelli seeks the ratio between the volumes of such solids and that of the sphere in which they are inscribed/circumscribed. In these works, there are several propositions whose proof is based on classical Euclidean methods without resorting to exhaustion, but many interesting statements, which concern the areas and the volumes of curvilinear or mixtilinear figures are proved by exhaustion. The first of this type of propositions is the third one, where Torricelli proves that given a right cylinder whose height:

$$AC = \frac{1}{4} AB$$

where AB is the diameter of the basis circle c (Fig. 1), the external surface of the cylinder is equal to that of c .

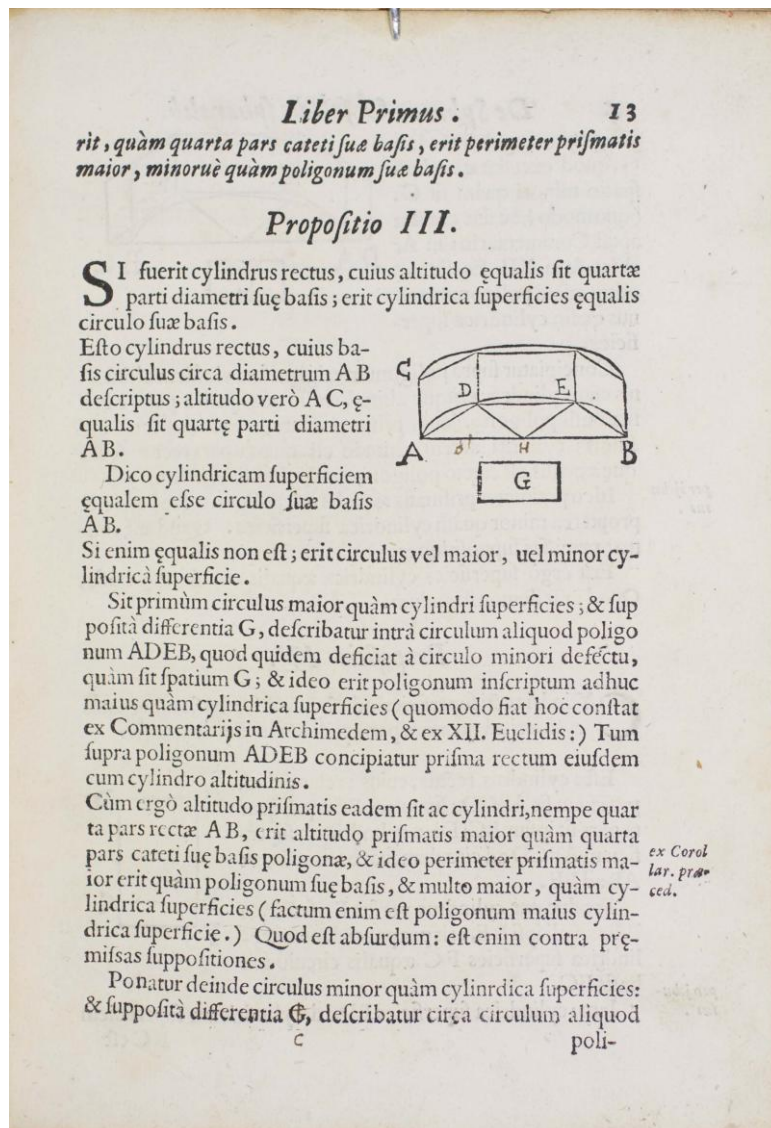


Fig. 1. The diagram used by Torricelli to prove Proposition III (Torricelli 1644, I, 1, p. 13; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 318-319)

Source: Courtesy University of Oklahoma, History of Science Collections

In the corollaries to his propositions, Torricelli highlights which theorems of *De Sphaera et Cyllindro* by Archimedes are deducible from the train of thought he was developing. This is a coherent confirmation of what Torricelli had claimed in his *Proemium*: his source of inspiration is indeed a work by Archimedes and his aim is to offer a more complete treatment than Archimedes's. The following proposition where exhaustion used is the eighth one. It states that, given a right cone, the area of its basis is to the area of the external surface as the basis' radius is to the side of the cone (that is the segment of a generatrix between the vertex and the basis circle). In the following propositions Torricelli offers a series of relations between the external surfaces of cones and cylinders.

Starting from Proposition XI, through a Lemma stated before such proposition, Torricelli introduced the truncated cone, also creating a new vocabulary for its significant elements. Thus (Fig. 2), given a truncated cone and a cutting plane perpendicular to the axis, the straight line EH which joins the mean points of the oblique sides of the trapezium $ABCD$ was called by Torricelli *Linea media aritmetica frusti conici* (arithmetical mean line of the truncated cone). The rectangle of dimension EB and AB was named *rectangulum proprium frusti conici* (proper rectangle of the truncated cone).



Fig. 2. The figure used by Torricelli to introduce the basic elements of the truncated cone (Torricelli 1644, I, 1, p. 20; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 332-333)
Source: Courtesy University of Oklahoma, History of Science Collections

An important result, obtained without resorting to exhaustion, is given in Proposition XII. Here Torricelli proves that the external surface of a truncated cone is is to that of a right cylinder as the proper rectangle of the truncated cone is to the rectangle having as dimensions the height of the cylinder and the diameter of the circle's basis. From this theorem, Proposition XVI of Archimedes's *De Sphaera et Cyindro* first book follows as a corollary: the external surface of a truncated cone is equal to the circle's whose radius is the mean proportional between the side of the truncated cone and its arithmetical mean. The first significant theorem, which concerns new rotation figures, is Proposition XIII (Fig. 3, particularly see p. 23). Here Torricelli considers two perpendicular diameters AB and CD of a circle and a tangent in a point I different from A, B, C, D . Produce the tangent so that I is the mean point of a segment which does not intersect AB produced. Then the external surface of the cylinder whose height is GH and whose basis circle has diameter equal to AB is the same as that of truncated cone whose side is the tangent segment ML and whose basis' radius is MN .

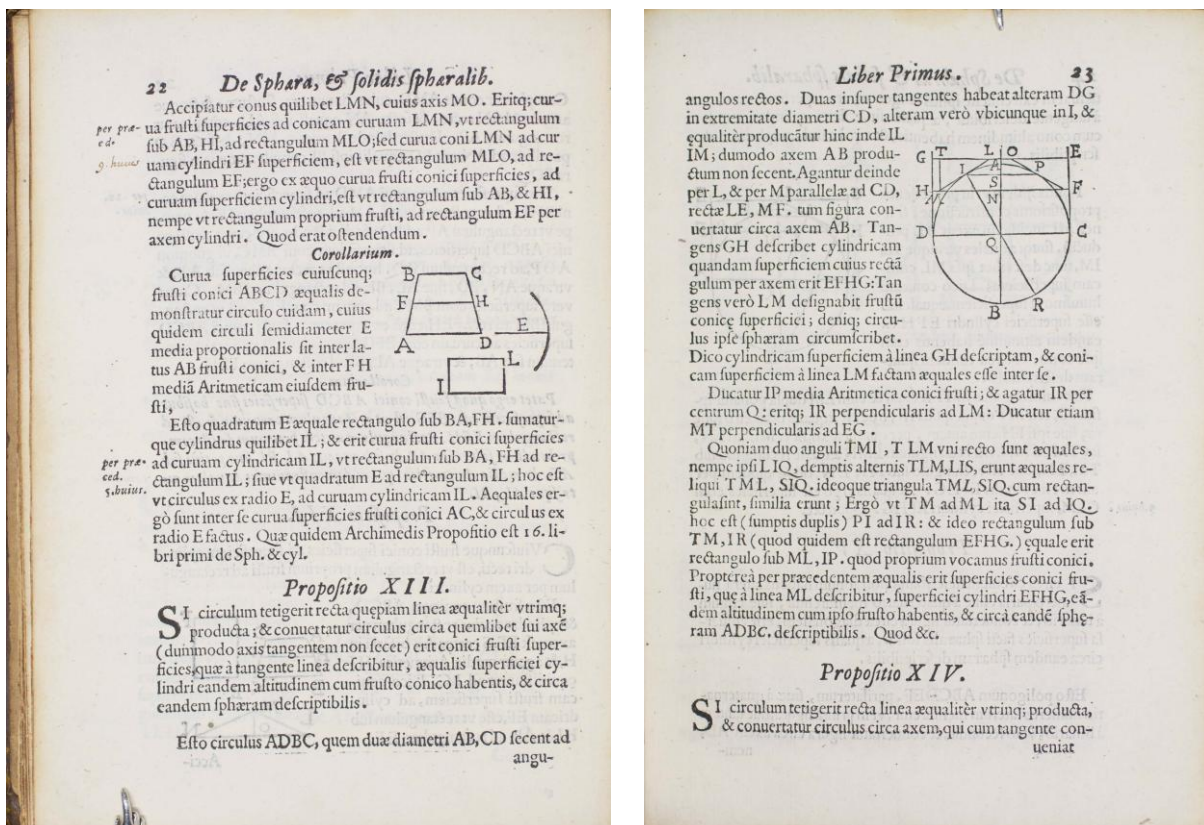


Fig. 3. The figure used by Torricelli to prove Proposition XIII (Torricelli 1644, I, 1, pp. 22-23; Pisano-Dhombres-Radelet de Grave-Bussotti, 2024, pp. 336-339)

Source: Courtesy University of Oklahoma, History of Science Collections.

The first proposition which concerns a “spherical solid” is the XVth one. Here Torricelli argues that, given a circle c and a circumscribed even-sided regular polygon, it rotates around its diagonal. The surface of the spherical solid so obtained is equal to the external surface of the cylinder having the same height as the spherical solid and circumscribed to the sphere, given by the rotation of c around a diameter.

Proposition XVI states the analogous theorem if the polygon rotates around its apothem. While the Proposition XVII considers the situation of a circumscribed odd-sided regular polygon rotating around its apothem.

Now Torricelli can prove (Proposition XVIII) that the surface of a hemisphere is equal to the external surface of a cylinder having the same basis and height. The beautiful and long demonstration is given by exhaustion.

Proposition XIX represents another magnificent example concerning the application of the exhaustion method. Here Torricelli proves that the surface of any spherical segment is equal to the

external surface of the cylinder circumscribed to the sphere and having the same height as the segment. Now Torricelli is ready to prove in two different manners – both different from Archimedes's – one of the most classical results proved in *De Sphaera et Cyindro*: the spherical surface is four times the surface of its great circle. The proofs are clear because they rely on the series of theorems previously proved by Torricelli. After that (Propositions XXIII and XXIV) Torricelli studies the external surface of a rhomboidal solid, that is a solid given by two cones having the same base circle and joined in that circle.

Among the immediately following propositions, the most interesting is Proposition XXVII where Torricelli proved: be given a truncated cone $ABCD$ (Fig. 4) and a cone BEC having the same basis as the bigger basis of $ABCD$ and its same height. The volume of the solid given by the difference of $ABCD$ and BEC is equal to that of the cone whose basis has the same surface as the external surface of the truncated cone and whose height is equal to the perpendicular EH elevated from the vertex of the cone to the side of the truncated cone.

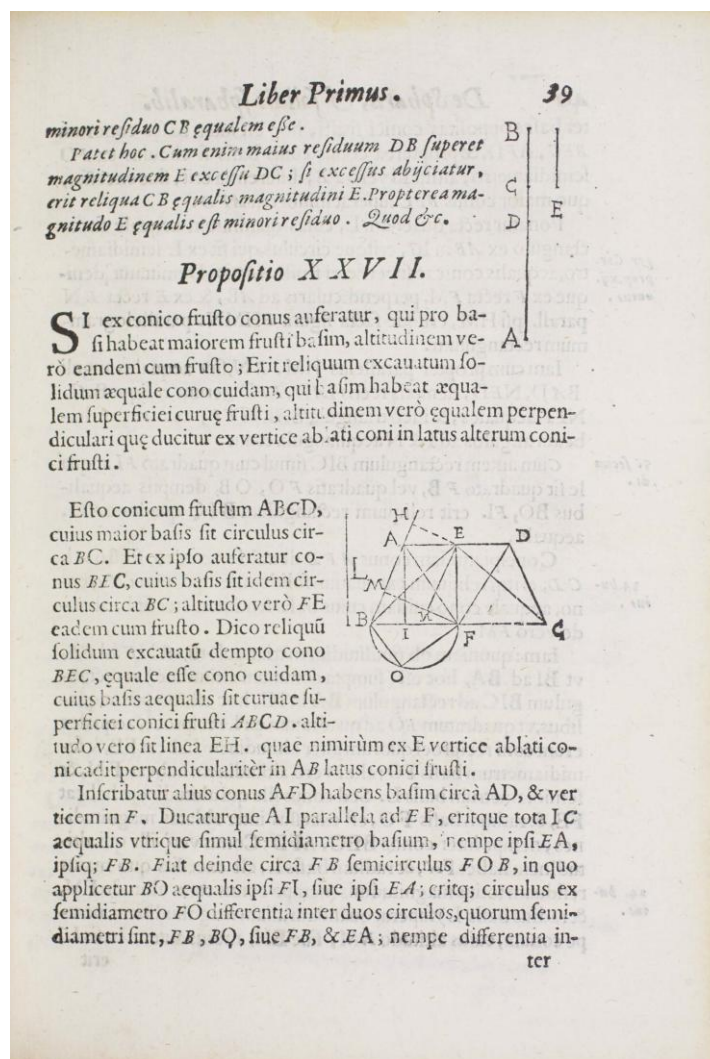


Fig 4. The figure used by Torricelli to prove Proposition XXVII (Torricelli 1644, I, 1, p. 39; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 370-371)
Source: Courtesy University of Oklahoma, History of Science Collections.

Torricelli offers two demonstrations of this theorem. In none of them is the exhaustion method used. These proofs, far from being trivial, are rather long and well argued.

Proposition XXIX is a beautiful example of infinitesimal geometry because Torricelli solves this problem: to inscribe in and to circumscribe to a revolution solid two figures composed of cylinders having the same height so that the difference between the volumes of the two figures is less than any given solid. Through this result, Torricelli proves as the thirtieth and last proposition of the first book, the classical Archimedes's result: the sphere has a quadruple volume in respect to the cone whose

basis is a great circle of the sphere and whose height is equal to the sphere's radius. The proof is based on the exhaustion method.

The first three propositions of the second book concern the volume of cones circumscribed to a sphere. For example, the first one states that, given a cone circumscribed to a sphere, its volume is equal to that of the cone whose base's area is equal to the whole surface of the previous cone and whose height is equal to the radius of the sphere.

Propositions IV-XIV regard the spherical solids produced by the rotation of an even-sided regular polygon. Let us consider the first two. In Proposition IV Torricelli proves that the volume of a spherical solid of second species is equal to that of a cone whose surface is equal to the surface of the solid and whose height is the sphere's radius. Torricelli clarifies that, when the number of sides is a multiple of 4, the proposition had already been proved by Archimedes in *De Sphaera et Cyindro*, while when the number of sides is of the form $4n+2$ the proof was missing. He furnished such a proof. Proposition V is dedicated to the spherical solid of fifth species. Torricelli proves that its volume is to the volume of the inscribed sphere as the surface of the solid is to the surface of the sphere. The same proportion (Proposition VI) holds between the diagonal of the solid and the diameter of the sphere.

From Proposition XV Torricelli begins to deal with odd-sided regular polygons. The first case analysed (Proposition XV) is that of a spherical solid of fourth species. Torricelli proved that its volume is equal to that of a cone whose basis is equal to the solid's surface and whose height coincides with the sphere's radius. The propositions until the XIXth one regard the solids generated by the rotation of odd-sided regular polygons. After that, a series of propositions begins where the volumes of the different spherical solids are compared. We will provide the reader with two examples.

In Proposition XX Torricelli proves that:

In a sphere of volume C , be described an even sided spherical solid of volume S generated by the rotation around the diagonal of the generating polygon. The sphere's volume is to $C-S$ in duplicate ratio of the sphere's diameter to the polygon's side.³⁰ (Fig. 5b).

³⁰ "Si intra sphaeram descriptum sit sphaerale solidum parilaterum, circumque diagonalem revolutum, erit sphaera ad excessum, quo ipsa solidum superat, in duplicata ratione diametri sphaerae ad latus poligoni". (Torricelli 1644, I, 2, p. 66; Authors' translation).

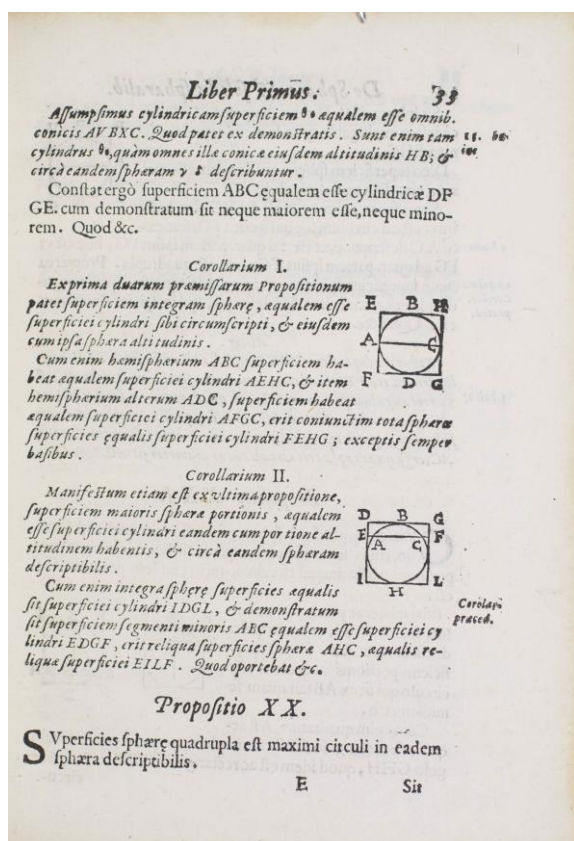


Fig. 5a. The Proposition XX. (Torricelli 1644, I, 1, p. 33; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 358-359)
Source: Courtesy University of Oklahoma, History of Science Collections.

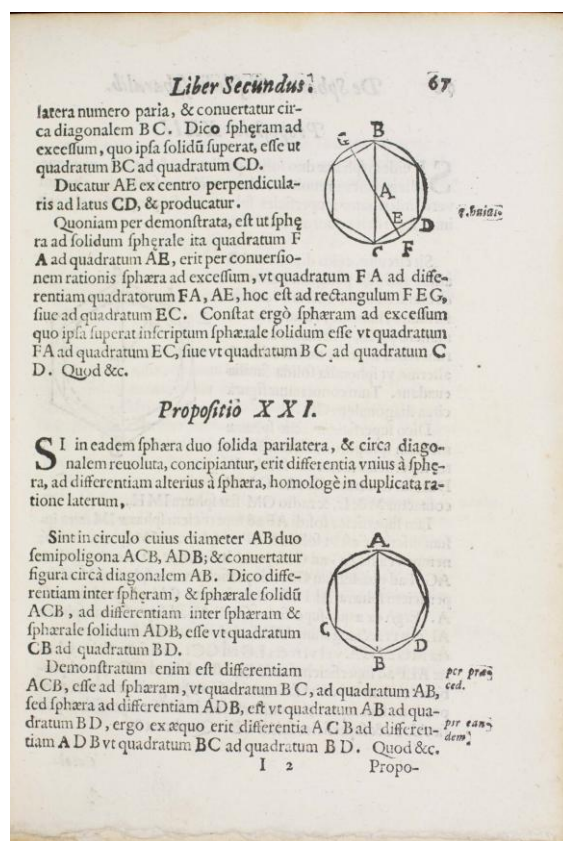


Fig. 5b. The diagram used by Torricelli to prove Proposition XX. (Torricelli 1644, I, 1, p. 67; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 426-427)
Source: Courtesy University of Oklahoma, History of Science Collections.

Torricelli proves that the sphere's volume is to the volume of the difference between the sphere and the inscribed solid as BC^2 is to CD^2 .

Proposition XXII is interesting because Torricelli compares the sphere's surface with those of an inscribed and a circumscribed even-sided sphaeral solid. This proposition states that given two similar even-sided solids, one inscribed in and the other circumscribed to the sphere, rotate these solids around their diagonal, the sphere's surface is mean proportional between the surfaces of the two solids.

The propositions that follow until the XXVIIth one are theorems on even-sided sphaeral solids. Their proofs are well argued.

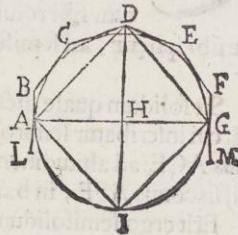
Proposition XXVIII (Fig. 6) is noteworthy. It concerns even-sided solids, but it distinguishes between those whose number of sides is of the form $4n$ and those whose number of sides is of the form $4n+2$. This theorem states that the volume of a sphaeral solid, generated by the rotation of a polygon, whose number of sides is of the form $4n+2$, around its diagonal is twice the solid-rhombus inscribed in the solid.

Liber Secundus. 73
erit vt quadratum ex CD ad duo quadrata ex AE. Quod &c.

Propositio XXVIII.

Quodlibet sphærale solidum circa diagonalem reuolutum (cuius latera numero quidem paria sint, sed nullo modo à quaternario mensurentur, vt sunt 6. 10. 14. 18. 22. &c.) inscripti sibi rombi solidi duplum est.

Sit solidum quale dictum est AB CDEFG. circa axē siue diagonalem DI reuolutum. Manifestū est quod duo latera opposita BL. FM. contingant sphæram in extremitatibus A, G, diametri AG, quæ quidem perpendicularis sit ad DI. quādoquidē laterum numerus à binario tantum mensuratur, non autem à quaternario.



Inscribantur iam duo conī; nempe ADG in semisolido, habens altitudinem HD.; conus verò AIG in hemisphærio. Erit igitur semisolidum ABCDEFG ad hemisphærium vt axis ad axem, nempe vt DH ad HI, hoc est vt conus ADG ad conum AIG (cum sint in eadem basi;) & permutando semisolidum ad suum conum ADG, erit vt hemisphærium ad suum conum AIG; quare duplum erit. Propterea omne solidum, quale dictum est duplum erit inscripti sibi rombi solidi, Quod &c.

Lemma.

Si hemisphærium ABC, & conus quicumque rectus DBE eandem altitudinem habuerint FB; erit hemisphærium ad prædictum conum vt duplum basis hemisphærij ad basim eiusdem conī.

Sit vt ponitur: Et inscribatur in hemisphærio conus ABC. Erit ergo conus ABC ad conum DBE vt basis AC ad basim DE;
K sum-

Fig. 6. The diagram used to Torricelli to prove Proposition 28. The even-sided solid is obtained by the rotation of the polygon DEFGI [...] around its diagonal DI and the solid-rhombus by the rotation of ADGI around the same diagonal. (Torricelli 1644, I, 1, p. 73; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 438-439)

Source: Courtesy University of Oklahoma, History of Science Collections.

Up to Proposition XXIXth Torricelli dealt with spherul solids. From Proposition XXX various theorems on the volumes of cones and cylinders inscribed in or circumscribed to a sphere are proved. With Proposition XXXV Torricelli begins to analyse the spherul solids generated by the rotation of a hexagon. For example, in Proposition XXXV he proves that the spherul solid's volume generated by the rotation of a hexagon around its apothem is seven times the cone's having the same basis and the same height. The reader can find specific and curious propositions, of clear demonstration, such as the following one (Proposition XLIII): given a regular hexagon, consider a second regular hexagon inscribed so that its vertices lie in the middle points of the circumscribed hexagon. Rotate the whole figure around the apothem of the circumscribed hexagon (and, hence around the diagonal of that inscribed). The volume of the circumscribed spherul solid is to that of the inscribed one as 14 to 9.

Among the remaining nine propositions XLIV-LII, the XLIVth one is particularly interesting. Torricelli compared the spherul solids generated by the rotation of a regular polygon around the diagonal and the apothem respectively. He proved that the volume of the first one is to that of the second one as twice the rectangle whose sides are the diagonal and the apothem is to the sum of the square of the diagonal and the square of the apothem. As a Corollary, it follows that the volume of the first solid is less than the other's. In the second book the method of exhaustion is never used. This work by Torricelli is written in the classical style of Greek geometry. In the *Proemium* Torricelli recognised that his source of inspiration was Archimedes. As a matter of fact, *De Sphaera et Solidis Sphaeralibus* might have been written by the Great Syracusan. Torricelli does not use any algebraic calculations. One could say that Archimedes in works such as the *Quadrature of parabolas* and *Conoids and spheroids* resorted to more algebraic calculations than did Torricelli in *De Sphaera et Solidis Sphaeralibus*! From a mathematical and historical perspective, now is the occasion to recall once again Archimedes' heritage in Torricelli. The latter used the method of exhaustion as rigorously as Archimedes and also extended Archimedes's results.

3.2 *De Motu Gravium Naturaliter Descendentium*

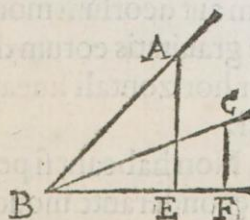
If Archimedes was the source of inspiration of *De Sphaera et Solidis Sphaeralibus*, Galileo inspired the second work inserted in *Opera Geometrica: De Motu Gravium Naturaliter Descendentium*, although the Archimedean influence is conspicuous in this work, too³¹. Specifically, the third and the fourth Days of *Discorsi e dimostrazioni intorno a due nuove scienze* are the crucial reference points of this writing, which is divided into two books: in the first one the fundamental elements of parabolic motion are studied; in the second the motion of projectiles is examined. It must be pointed out that not all physicists were convinced that the trajectory of a projectile was a parabola. Among other purposes, Torricelli also aimed to show it was a parabola. In his introductory considerations, the author clarifies that his book was written for specialists who were familiar with Galileo's works. The fundamental concept was that of "*momento*" (*momentum*). In fact, it had been introduced by Benedetti, but studied systematically by Galileo. One might say that the momentum (Galluzzi 1979) of a weight is the measure of weight's effect in function of the position of the weight itself. The fundamental theorem is that stated first by Galileo as the sixth proposition of his *Le Mecaniche* which claims that the *momenta* of equal weights on differently inclined planes are as the perpendiculars drawn from the weights to the Earth's surface if the length of the planes where the weights lie is the same (Fig. 7).

³¹ With regard to the influence exerted by Archimedes on Torricelli's *De Motu Gravium*, see Pisano 2009a; see also: 2009b, 2009c, 2009d; Pisano and Capecchi 2010a, 2010b and Pisano and Bussotti 2014b, 2014a; Pisano 2001.

demonstratione confirmabimus: protinus ad ostendendum id quod Galileo principium sine petito est, accedemus.

Premittimus.

Duo graua simul coniuncta ex se moueri non posse, nisi centrum commune grauitatis ipsorum descendat.

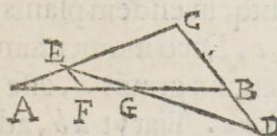


Quando enim duo graua ita inter se coniuncta fuerint, ut ad motum unius motus etiam alterius consequatur, erunt duo illa graua tamquam graue unum ex duobus compositum, siue id libra fiat, siue motica, siue qualibet alia Mechanica ratione, graue autem huiusmodi non mouebitur unquam, nisi centrum grauitatis ipsius descendat. Quando vero ita constitutum fuerit ut nullo modo commune ipsius centrum grauitatis descendere possit, graue penitus in sua positione quiescet: alias enim frustra moueretur; horizontali, scilicet latrone, quae nequaquam deorsum tendit.

PROPOSITIO I.

SI in planis inaequaliter inclinatis, eandem tamen eleuationem habentibus, duo graua constituentur, quae inter se eandem homologam rationem habeant quam habent longitudines planorum, graua aequale momentum habebunt.

Sit ab . horizon, & plana inaequaliter inclinata ca , cb . Fiat ut ac ad cb , ita graue aliquod a . ad graue b . Et graua haec in homologis planis collocentur, in punctis a , & b . eiusdem horizontalis lineae. Connectantur etiam aliquo imaginario funiculo per a cb . ducto, adeo ut ad motum unius motus alterius consequatur.



Dico graua sic disposita aequale momentum habere: hoc est

N 2

in ea

Fig. 7. The fundamental theorem claims that, if $AB=BC$ and if in A and C two equal weights a and c lie, the momentum of a is to the momentum of c as AE is to CF . (Torricelli 1644, II, 1, pp. 99; see figure top right side; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 490-491)

Source: Courtesy University of Oklahoma, History of Science Collections.

Galileo presented this proposition as a given. Torricelli was going to prove it.

The *De Motu Gravium Naturaliter Descendentium* has to be regarded as an original contribution to the development of physics because Torricelli saw the seed of the whole science of movement in his famous principle:

Two heavy bodies joined together cannot move by themselves unless their common centre of gravity descends³².

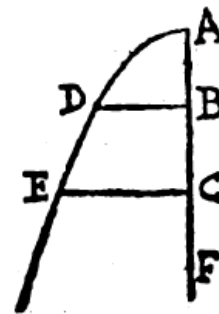
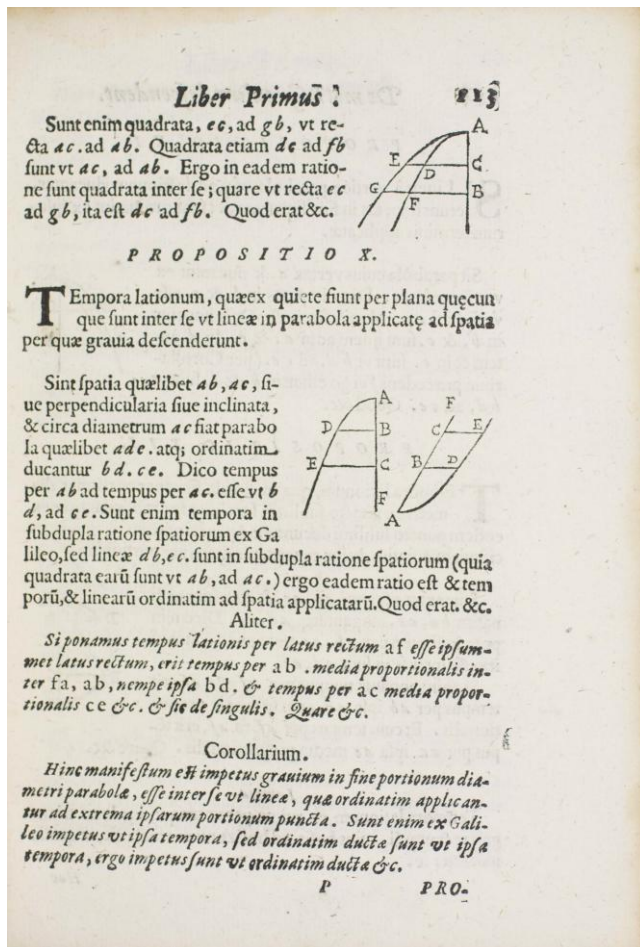
Torricelli's idea is, thus, an element of the physics of principles which, at the end of the 17th century, was by far overtaken by Newton's physics of forces, but that, from the second half of the 18th century, became a fundamental way to face the problems of this discipline (Cfr. Pisano and Capecchi 2013). Torricelli's principle is directly tied to that of virtual velocities (Cfr. Pisano and Bussotti 2016, 2017b).

In the initial propositions of his work, Torricelli analyses the momenta of spheres or systems of spheres posed in different positions on an inclined plane. As a matter of fact, he distinguishes the heavy body's momentum in a total momentum and in its components determined by the sinus and the cosine of the angle of inclination of the inclined plane in which the heavy body is posed.

In Proposition III Torricelli proved the fundamental theorem by Galileo that the moments of two equally heavy bodies lying on two equally long, but unequally inclined planes, are as the perpendiculars from the highest vertex of the plane to the ground (Torricelli 1644, p. 104). In the proof, Torricelli's principle plays an indirect role because it is not exploited explicitly. However, Proposition II is used. The proof of this proposition relies upon Proposition I, whose demonstration is based on the principle. A series of propositions, which have a clear Galilean flavour, follows. For example, in Proposition IV, Torricelli proves that, given two inclined planes of the same altitude, the fall-times are as the planes' length. As Torricelli himself recognizes, this proposition is a consequence of Galileo's famous Theorem VI, Proposition VI of the third Day of *Discorsi e dimostrazioni*: given a vertical circle, if, from its highest point you draw any number of inclined planes so that their lengths cut the circumference, the fall-times along these planes are equal.

Starting from Proposition VI some clear properties of the parabola and of the representations of motions through parabolas are introduced. For example, in Proposition X, Torricelli proves that, given a body starting from rest and falling on inclined planes, the fall-time is as the straight lines applied to the parabolas in the considered point (Fig. 8).

³² "Duo gravia simul coniuncta ex se moveri non posse, nisi centrum commune gravitatis ipsorum descendat". (Torricelli 1644, II, 1, p. 99; Authors' translation). On the state of the art before Torricelli's centre of gravity, see Pisano 2007, 2009a, 2009b, 2009c, 2009d, 2011; Capecchi and Pisano 2007a, 2007b, 2007c, 2008, 2010a, 2010b, Bascelli 2010. For an examination of this principle see below Pisano and Bussotti' article in this book.



Consider a parabola ADE . Be AF the fall-height. The fall-time from A to B is to the fall-time from A to C as BD is to EC .

Fig. 8. The figure that Torricelli used to prove Proposition X

(Torricelli 1644, II, 1, p. 113; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 518-519)

Source: Courtesy University of Oklahoma, History of Science Collections.

Many of these propositions by Torricelli are based on Galileo's theorem that, in free fall, spaces are inversely as the time squared. It is remarkable that some scientists, for example Descartes and Roberval, were not convinced that Galileo's law was generally correct. Roberval on the 1st of January 1646 wrote to Torricelli that he did not believe in Galileo's laws and put forward some arguments. In his answer (1646, July 7th) Torricelli claimed, in substance, that Galileo's and his own wider considerations are correct at least from a mathematical point of view, that is: supposing Galileo's law to be true, then the consequences explored by Galileo and more broadly by Torricelli himself are true. If from a certain perspective, Torricelli's answer is undoubtedly disappointing because it is not an answer to Roberval's doubts, from another perspective, it is interesting. For, Torricelli seems interested in rational mechanics, almost independently of physical applications, whereas Roberval is wondering how nature behaves; namely in physics rather than in rational mechanics. With Proposition XVII the study of parabolic motion begins. This proposition is a problem (fig. 9): given a parabola of vertex B , find the point A such that, if a body falls from A and, arrived in B with a certain *impetus*, such *impetus* is directed horizontally, the body's trajectory is the parabola.

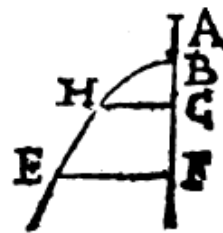
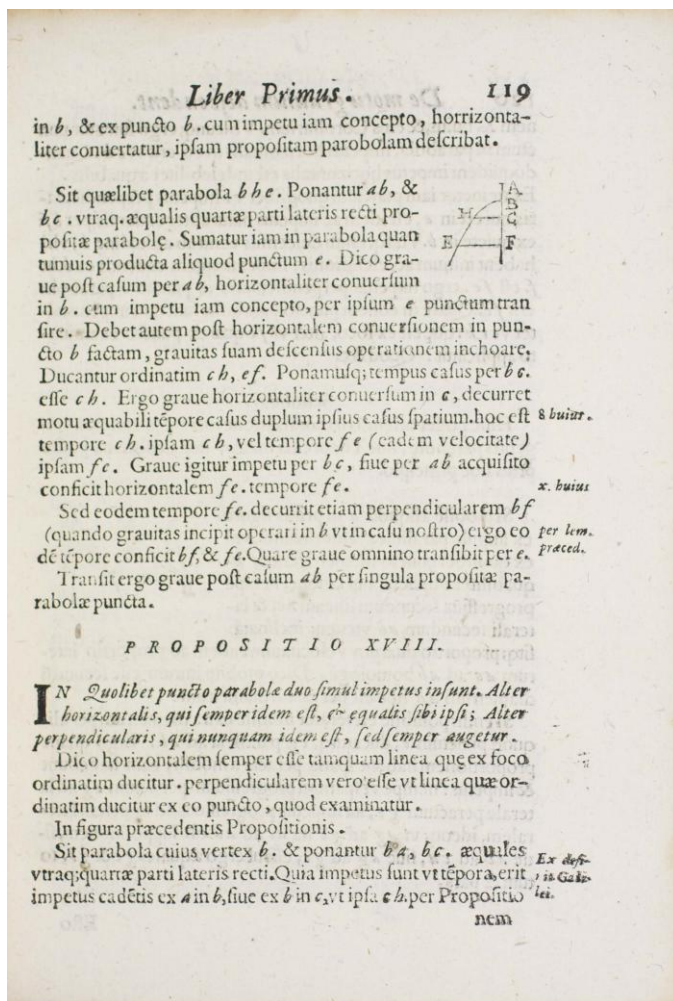


Fig. 9. The figure used by Torricelli to solve the problem expounded in Proposition XVII (Torricelli 1644, II, 1, p. 119; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 530-531)
Source: Courtesy University of Oklahoma, History of Science Collections.

It must be noted that the term *impetus* does not have a single precise meaning. In this case it means velocity.

In the following proposition, the classical Galilean theorem that in a parabolic motion the horizontal component of velocity is constant, while the vertical is uniformly accelerated is proved.

In the noteworthy scholium following Proposition XVIII, the parallelogram rule for velocities is proved by a remarkable *ad absurdum* reasoning. In the same scholium Torricelli teaches how to draw the tangent at a point of a parabola by means of kinematic considerations. Through a similar reasoning he also explains how to draw the tangent to an Archimedean spiral and also outlines the possibility to draw the tangent at a point of the cycloid. The solution of this last problem caused the priority dispute with Roberval. A long series of propositions follows in which several aspects of the motion along a parabola are analysed: velocities, times, etc. Torricelli also examines the case in which a body falls along the chords of a parabola and proves (Proposition XXVII, Fig. 10) that the fall-time along the chord of a parabola having an extreme in the vertex is directly as the length of the chord and inversely as the ordinate of the second extreme of the chord itself.

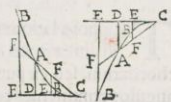
Est enim bc . ad di longitudine ut ca . ad ad . vel bc ad de . potentia. Quare in continua proportionem sunt bc . de . di & c . & de media est, Quod oportebat ostendere.



PROPOSITIO XXVI.

SI Horizontalis linea parabolam contingat, tempora casuum ex punctis parabolę vsq; ad horizontę; siue ex punctis horizontis, ut in secunda figura, vsq; ad parabolam; erunt ut lineę parallele inter horizontem, & quamlibet aliam ex contactu inclinam interceptę.

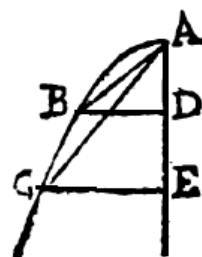
Sit parabola bac . cuius uertice c & eam in c tangat horizon cd . & ex contactu c . inclinetur utcumq; ca ducaturq; ex puncto a . ad horizon ti perpendicularis. Ducta iam quamlibet perpendiculari ef . Dico tempus casus per eb . esse fe . & sic de singulis. Ponamus tempus per ad . esse ad . erit tempus per be . ipsa fe . media proportionalis uti demonstratum est. Quare &c.



PROPOSITIO XXVII.

Tempora lationum per chordas ex uertice parabolę inclinatę, compositam rationem habent, ex ratione longitudinum chordarum, & ex ratione ordinatim applicatarum, contrarię tamen sumptarum.

Sint chordę ex uertice ab , ac . & ordinatim ducantur bd , ce . Dico tempus per ab . ad tempus per ac . habere rationem compositam ex ratione ab ad ac , & ex ratione c ad bd . Si enim concipiamus lationes il-



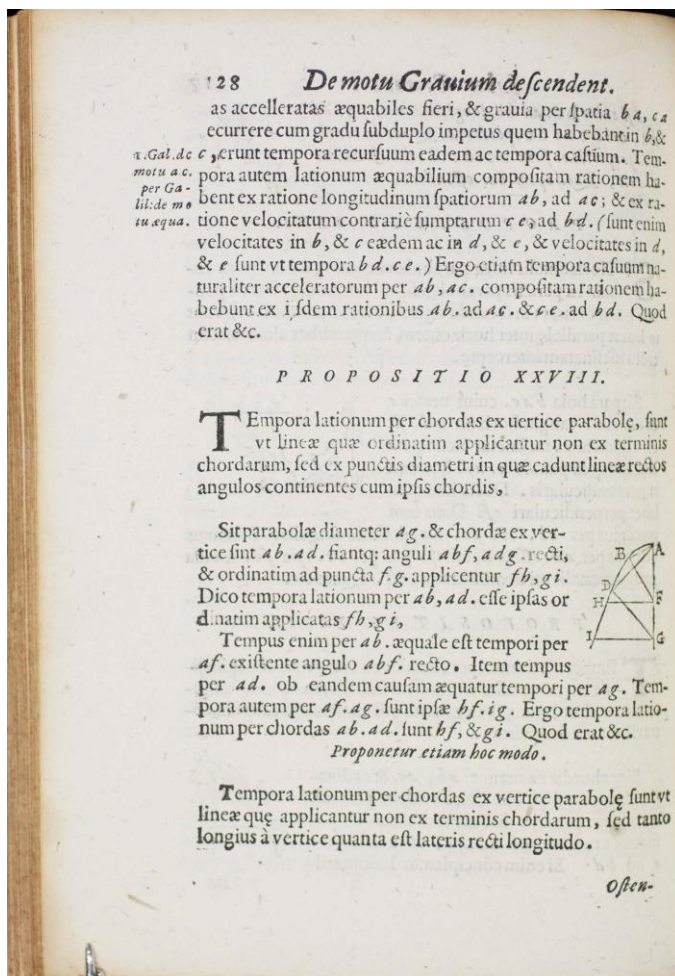
Consider a body falling along the chord AB and a body along the chord AC . Their fall-times are as $\frac{AB}{AG} \cdot \frac{CE}{BD}$.

Fig. 10. The diagram used by Torricelli to prove the Proposition XXVII

(Torricelli 1644, II, 1, p. 127; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 546-547)

Source: Courtesy University of Oklahoma, History of Science Collections.

Torricelli proves (Proposition XXVIII) that it is also possible to find a single segment which is as the fall-time along a chord. For, given the chord AB , it is enough to draw the straight line perpendicular to AB . Such a line will cut the parabola's axis in a point F . Draw the ordinate FH at the parabola in F . The fall-time along AB will be as FH (Fig. 11).



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as acceleratas æquabiles fieri, & gravia per spatia ba , ca ecurrere cum gradu subduplo impetus quem habebant in b , & c , erunt tempora recursum eadem ac tempora casuum. Tempora autem lationum æquabilium compositam rationem habent ex ratione longitudinum spatiorum ab , ad ac ; & ex ratione velocitatum contrariè sumptarum ce , ad bd . (sunt enim velocitates in b , & c eadem ac in d , & e , & velocitates in d , & e sunt vt tempora bd , ce .) Ergo etiam tempora casuum naturaliter acceleratorum per ab , ac , compositam rationem habebunt ex iisdem rationibus ab , ad ac , & ce , ad bd . Quod erat &c.

PROPOSITIO XXVIII.

Tempora lationum per chordas ex vertice parabole, sunt vt lineæ quæ ordinatim applicantur non ex terminis chordarum, sed ex punctis diametri in quæ cadunt lineæ rectos angulos continentes cum ipsis chordis.

Sit parabole diameter ag , & chordæ ex vertice sint ab , ad , fiantq; anguli abf , adg recti, & ordinatim ad puncta f , g , applicentur fb , gi . Dico tempora lationum per ab , ad , esse ipsas ordinatim applicatas fb , gi .

Tempus enim per ab , æquale est tempori per af , existente angulo abf , recto. Item tempus per ad , ob eandem causam æquatur tempori per ag . Tempora autem per af , ag , sunt ipsæ bf , ig . Ergo tempora lationum per chordas ab , ad , sunt bf , & gi . Quod erat &c.

Proponetur etiam hoc modo.

Tempora lationum per chordas ex vertice parabole sunt vt lineæ quæ applicantur non ex terminis chordarum, sed tanto longius à vertice quanta est lateris recti longitudo.

Osten-

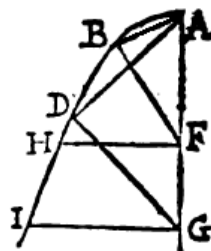


Fig. 11. The diagram used by Torricelli to prove Proposition XXVIII (Torricelli 1644, II, 1, p. 128; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 548-549)
Source: Courtesy University of Oklahoma, History of Science Collections.

From Proposition XXIX a series of theorems and problems are presented which concern the free fall along inclined planes. Most of these propositions are connected to the fundamental statement by Galileo on the fall-time along a chord of a circle. For example, in Proposition XXXV Torricelli considers a perpendicular of given altitude on the horizon and a horizontal segment a of which one extreme is the foot of the perpendicular. It is required to find an inclined plane, along which the fall time is the same as along the perpendicular. An obviously necessary and sufficient condition for the problem to have a solution is that a is not longer than half the perpendicular.

The three propositions XLI-XLIII are a very interesting digression (as Torricelli himself claims) on the motion along a hyperbola and an ellipsis.

In Proposition XLI Torricelli considers a horizontal segment AB divided into four equal parts AC , CD , DE , EB . From C and E he draws a hyperbola whose foci are A and B respectively (Fig. 12). Consider a point F on a branch FE of the hyperbola. The fall time along FA is the same as the fall time along FB plus the time in which the segment AB is traversed with the speed acquired in B .

Liber Primus

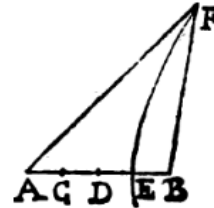
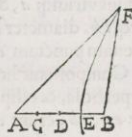
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circa hyperbolem quasdam nugas meditatam fuisse ad motum spectantes. Si cui conica non placent, digressionem hac prætermittam, paucâ hac enitare poterit, Et ad Propositionem 44. se conferre. Materia præcedentium hanc continuationem nimis expositabat.

PROPOSITIO XLI.

Si recta linea ab . in quatuor æquales partes ac, cd, de, eb . divisâ fuerit, & ex punctis c, e , excitentur duæ hyperbolæ, quæ sectiones oppositæ dicuntur, quarum foci sint a, b . Sumpto in altera earum quolibet puncto f . erit tempus per fa , æquale tempori per fb, ba .

Hoc enim patet ex præcedentibus. Nam propter hyperbolam linea fa , æqualis est ipsi fb, ce . per 5. i. tertij Conicorum. Sed ce . semilis est spatij horizontalis ab . per hypotensim, ergo æqualia sunt tempora lationum tam per fa , quam per fb, ba . Quod erat &c.



PROPOSITIO XLII.

Si datum sit horizontale spatium ab terminatum, & longitudo alicuius plani f . data sit maior quam ab . Secare oportet planum f . in duas partes inæquales ea lege, ut si ex terminis a, b , facta plana ad idem punctum componentur, tempus lationis per maius planum, æquale sit tempori lationis per minus planum & per horizontalem simul.

Hoc duplici modo absoluemus. Primum contemplatiuè, siue per resolutionem, deinde practicè.

Resoluiuè hoc modo. factum iam sit quod faciendum est. & sint duo plana ea, eb , ut imperatum est, nempe æqualia simul

S
mul

Fig. 12. The diagram used by Torricelli to prove Proposition XLI.

(Torricelli 1644, II, 1, p. 137; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 566-567)

Source: Courtesy University of Oklahoma, History of Science Collections.

Proposition XLII is a fascinating problem (Fig. 13): be given the horizontal segment AB and the segment F longer than AB . It is required to cut F in two unequal sections so as to form an inclined plane whose basis is AB and whose vertex is E such that:

- 1) $AE + EB = F$;
- 2) The fall time along the longer side of the plane AE is equal to the fall time along EB plus the time necessary to traverse the segment AB with the velocity acquired in B .

Torricelli proves that E is the intersection point between the ellipsis whose foci are A and B and whose major axis is $CD = F$ and the hyperbola whose foci are A and B and whose vertex is in the point H obtained by dividing into four equal parts the segment AB . Thus, it is

$$BH = \frac{1}{4} AB.$$

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Dico tempora per ib , & per lh . æqualia esse.

Concipiamus basim ce . trianguli cde , esse ad horizontem erectam. Manifestum est tempora lationum per cd , de . equalia esse. per lemma Propof. 13. Sed cum duo latera cd , df . duobus lateribus ib . hg . æqualia sint vtrinq; vtrique, & anguli efd , igh . recti, si ex quadratis equalibus dc . hi . demantur quadrata æqualia df . hg . remanebunt æqualia quadrata fc . gi , & ideo fc . gi . lineæ æquales erunt; & propterea integra triangula cdf . ihg . æqualia, & similia erunt. & tempus per ib . æquale tempori per cd .

Eodem modo ostendetur tempus per lh . æquale tempori per de . Quare cum æqualia sint tempora per cd . de . æqualia erunt etiam per ib . hl . Quod erat &c.

PROPOSITIO II.

Si ex puncto sublimiori circuli ad horizontem erecti graue cadat vsque in centrum b , & inde per quodcumq; planum siue eleuatum, siue decliue, conuertatur cum impetu iam concepto; graue huiusmodi tempore casus ab absoluet spatium bd ; quod nempe æquale sit vtrifq. tum semidiametro bc , tum etiã ipsius perpendiculari ce .

Secũta sit cd equalis ipsi ce . Dico tempus per ab , ex quiete in a , æquale esse tempori per bd . post casum ab . Est. n. ob equalitatem vt dc ad cb , ita ce ad ba , hoc est cf ad fb , & permutando vt cd ad cf , ita cb ad bf . in vtraq. figura.

Sed in prima tantũ erit componendo vt df ad cf , ita cf ad bf . In secunda verò erit. Conuertendo, per conuertionẽ rationis, & iterum conuertendo, vt df ad cf , ita cf ad bf . Quare in vtroq. casu tres lineę df , cf , bf . sunt in continuã proportionẽ.

Iam si tempus per ab ponatur esse ab , erit tempus per fb ip
sa fb .

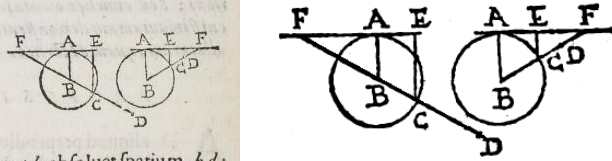


Fig. 14. The figure used by Torricelli to prove the Proposition II. On the left, the plane is inclined downwards, on the right upwards (Torricelli 1644, II, 1, p. 144; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 580-581) Source: Courtesy University of Oklahoma, History of Science Collections

Torricelli offered two different demonstrations of this theorem.

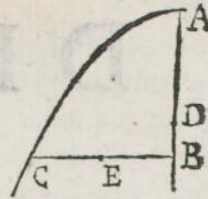
Problem LV concerns an interesting question of minimum. Torricelli considers as given a horizontal segment AB . Be also given the perpendicular in A (or in B) at AB . On such perpendicular, determine a point C such that the time to traverse $CA+AB$ (or $CB+BA$) is the shortest. The answer is that CA (or CB)= $1/2AB$. It must be noted that, in this case, too, the Galilean character of Torricelli's works is apparent, because Galileo in the third Day of *Discorsi e dimostrazioni* had also faced problems of minimal time, for example, in the theorems XIX, XX (Propositions XXX and XXXI respectively).

The LVIIIth and last proposition of the first book proves an interesting property of parabolic motion (Fig. 15): consider a segment AB of the parabola's axis and the time necessary for a body in free fall to traverse AB plus the ordinate BC with the speed acquired in B . This time is equal to that necessary for the body in free fall to traverse a segment DB as long as $1/4$ of the parabola's *latum rectum* plus the ordinate BC with the speed acquired in B falling from D .

Liber Primus.

153

Sit axis parabolæ, ab , eius ordinatim applicata be . Et secetur bd . æqualis quartæ parti lateris recti. Dico tempora per abc . & per dbc . æqualia esse inter se. Diuidatur bc . bifariam in e .



Erunt tum quadratum eb , tum rectangulum abd , subquadrupla quadrati cb , Sunt ideo æqualia inter se, & ipsa eb . media proportionalis est inter ab , bd . Quare per Corollarium primum Propositionis 50. huius, tempora per abc . & per dbc æqualia sunt &c.

Sunt etiam per secundum eiusdem Propositionis Corollarium, reciproce æqualia. Quod satis sit ostendisse circa motum grauium naturaliter descendentium.

Finis Primi Libri.



V

DE MO

Fig. 15. The figure used by Torricelli to prove Proposition LVIII (Torricelli 1644, II, 1, p. 153; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 598-599) Source: Courtesy University of Oklahoma, History of Science Collections.

The first book of this work is conceived having the third Day of Galileo's *Discorsi e dimostrazioni* as its source of inspiration. This notwithstanding, there are many original ideas, foremost the Torricelli

principle and its use. Some specific propositions, as those on the movement in conic sections different from a parabola, or the last ones we have referred to, are interesting and show that Torricelli had the intention to establish the science of mechanics more rigorously and broadly than Galileo, though remaining in the line of reasoning indicated by Galileo.

It seems to us necessary to mention at least one consideration developed by Belloni: in a letter to Mersenne written in January 1645 Torricelli expresses the conviction in the existence of a proportionality between the cause of motion's change (*momentum*) and velocity³³. This means that, as it were, Torricelli's – like Kepler's forces – produce velocities and not accelerations, which, in its turn, indicates that the inertia principle was not yet grasped in all its features.

The second book of *De motu gravium naturaliter descendendum* concerns the projectiles' motion and is inspired by the fourth Day of Galileo's *Discorsi e dimostrazioni*. Torricelli's aim is stated in his introductory considerations: in the Corollary of Theorem IV, Proposition VII of *Discorsi*'s fourth Day, Galileo stated that the trajectory of a projectile is a parabola, exactly as the motion of a body to which a horizontal (direction) velocity is impressed. Torricelli firmly believed in the truth of this proposition, but thought that so far it had been more supposed true than really demonstrated. Thus, his first aim is to offer a proof of this property. Afterwards, several characteristics of the motion of projectiles will be analysed relying upon such theorem, which Torricelli proves as Proposition III.

The Definition with which the book begins is interesting:

The direction of projection is defined as that straight line which touches the curved line of the projectile in the first point of that curved line itself [...].³⁴

Here there is the concept of tangent at a curve in its initial point, which is implicitly associated with the idea of an incipient motion. It is surely premature to speak of concepts close to those of infinitesimal analysis, but Torricelli's notion of tangent in the first point of the curve is interesting. The first remarkable proposition is the second one. Here Torricelli proves that the trajectory of a projectile in its highest point *B* cuts the perpendicular *BH* intercepted between the horizon and the line of direction into two equal parts *BC* and *CH* (Fig. 16).

The geometrical setting of Proposition II is illustrated in Fig. 16.

³³ Belloni adds a note to his Italian translation of *De motu gravium naturaliter descendendum* (Torricelli 1975, pp. 164-166, note 7; Authors' translation) where the Italian translation of Torricelli's letter to Mersenne is referred to.

³⁴ "Directio projectionis dicitur linea recta quae tangit lineam curvam proiecti in primo punto eiusdem lineae curvae [...]" (Torricelli 1644, II, 2, p. 155; Authors' translation).

idemq; gradus impetus eodem tempore; motuq; æquabili perducit mobile per duo spatia inæqualia ac , ad . Quod est absurdum.

PROPOSITIO II.

Semita proietorum, quæcunq; illa sit, sublimiori sui puncto bifariam secat perpendicularum quod inter horizontem, & lineam directionis interceptitur.

Proiciatur mobile ex a iuxta directionem utcumq; eleuatam ab . Patet quod sine tractione gravitatis procederet mobile motu recto, & æquabili per lineam directionis a b . Sed gravitate intus operante ab ipsa directione statim declinare incipiet, crescente semper deviationis mensura; & describet aliquam lineam curvam acd , quæcunq; sit. Hac linea punctum aliquod sublimius cæteris habet; illud nempe quod est ascensionis extremum, & primum descensionis. Sit huiusmodi punctum c , & per c ducatur perpendicularum bch . Dico hb , duplam esse ipsius bc .

Abstrahamus motum horizontalem; hic enim motus, quo ad lationem perpendicularem de qua agemus est tamquam non esset; cum illam neq; iuvet, neq; impediat. Concipiamus etiam mobile habere semper secum suum perpendicularum hb , horizontali quadam latione una cum ipso translata ex a versus b ; in quo perpendiculo ascendit graue, motu quodam continuo, sed semper magis ac magis deficiente, à puncto b vsq; ad punctum c . Conficit ergo mobile in suo perpendiculo tempore Exempli gratia e . spatium bc . sed si motu æquabili ascendisset cum impetu & tempore eodem, reperiretur in b (deberet enim ob motum æquabilem esse semper in communi sectione linearum ab , hb .) Quare per præcedentem, spatium hb ipsius bc duplum est. Quod erat &c.

PRO-

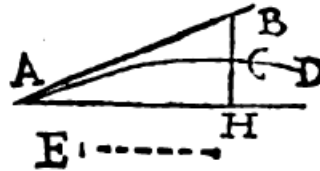


Fig. 16. The diagram used by Torricelli to prove proposition II. (Torricelli 1644, II, 2, p. 156; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 604-605) Source: Courtesy University of Oklahoma, History of Science Collections.

At the beginning of the demonstration, Torricelli claims that, without gravity, the projectile would continue its motion uniformly along the projection line. Belloni claims this to be a clear indication that Torricelli had a broader conception of inertia than Galileo because Galileo recognised only horizontal inertia (Torricelli 1975, p. 224, note 3). Without confronting now one of the most problematic questions of the whole history of science, namely what Galileo's concept of inertia was, it must be noted how elusive and difficult the notion of inertia is. It is true that Torricelli probably had a broader conception of rectilinear inertia than Galileo, but the fact that he believed a force to produce a velocity rather than a change of velocity, is a sign that a full comprehension of this concept had not yet been achieved. Proposition III is the fundamental statement which is at the basis of the further acquisitions explained in the book:

The curved line described by a mobile thrown with any inclination is a parabola. It is that parabola which the mobile would describe if it were thrown horizontally from the vertex of the curve.³⁵

Torricelli developed a long and detailed proof of this proposition, so that no doubt of its truth could remain. The propositions until the eight one can be interpreted as corollaries of the third one. The Proposition IX solves, instead, a significant problem.

³⁵ "Linea curva, quæ describitur a mobile secundum quamlibet elevationem proieto, parabola est, et prorsus, eadem, quam describeret mobile si cum horizontali impetu proiceretur a vertice eiusdem lineæ curvæ." (Torricelli 1644, p. 157; Authors' translation).

Be given the impetus BA , namely according to the definition, that acquired by a body which falls naturally from B to A . Be given the throw-direction AI and be the throw-speed that acquired in A . It is required to find the length, the height and the whole route of the parabola³⁶.

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Liber Secundus.

sub certam & uniuersalem mensuræ regulam cadere nō potest. Exempli gratia. Quando dicimus sit impetus datus ab. tunc sensus noster fert. Sit impetus datus tantus quantus requiritur ad proiciendum mobile ex a vsq; ad summum punctum perpendiculari b. Sive, quod idem est, quantus est impetus naturaliter cadentis ex b vsq; in a.

B
A

Lemma.

Circa diametrum ab. per verticem a, & quoduis punctum c. alia, atq; alia parabola non constituitur. Si enim possibile est, sint circa diametrum ab per puncta a & c. due parabole, & ex c ducatur ordinatim cb. tum alia quælibet ordinetur df. Quadratum ergo cb, ad duo illa quadrata inæqualia fd, se eandem rationem habet, nempe eam quam habet ba, ad af. Quod est absurdum. Ergo circa diametrum &c.

A
B
C
D
E
F

P R O P O S I T I O I X.

Dato impetu *ba* (hoc est quantus est naturaliter cadentis ex *b* in *a* iuxta definitionem) dataq. directione *az*, iuxta quam facienda proiecitur cum impetu dato. Oportet altitudinem, altitudinem, totamq; futuram parabolam huius proiectionis reperire.

Ducantur per a & b horizontales linee ad. bl. & fiat semicirculus asfb circa diametrum ab, qui lineam ac omnino secabit, cum ipsa ad tangens sit. Secetur in f, & ducatur fe horizontalis, & producat f g equalis ipsi fe. demum agatur per g perpendicularum l g d. Fiat iam circa diametrum gd per puncta g & a. parabola ag, quæ vnica erit per lemma præcedens. neq; alia parabola circa diametrum gd per puncta g, & a fieri poterit. Dico hanc esse parabolam quam-

B
I
L
E
A
D
C
F
G

tam.

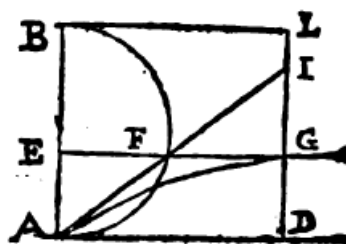


Fig 17. The diagram used by Torricelli to prove Proposition IX (Torricelli 1644, II, 2, p. 165; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 622-623) Source: Courtesy University of Oklahoma, History of Science Collections

After this problem, a series of propositions follows in which, given two data (for example *impetus* and length, Proposition XI), it is required to find other data (i.e., height, Proposition XI). Torricelli also discusses the cases in which the proposed problem cannot be solved. His work continues analysing the properties of maximum and minimum of a throw whose direction is an angle of 45 degrees (Prop. 17). A series of propositions on the throw-time follows. All the propositions from the Xth to the XXth are of clear demonstration. Propositions XXI and XXII solve, instead, two more

³⁶ “Dato impetu *ba* (hoc est quantus est naturaliter cadentis ex *b* in *a* iuxta definitionem) dataque directionem *ai*, iuxta quam facienda proectio cum impetu dato. Oportet amplitudinem altitudinem totamque futuram parabolam huius projectionis reperire.” (Torricelli 1644, p. 165; Authors' translation).

difficult problems: in the former Torricelli requires finding the constant horizontal velocity (*impetus*) as well as the variable vertical velocity in any point of the parabolic trajectory, given the initial velocity and the maximal height of the throw. In this proposition as well as in others, the throw-speed is not given directly but through the concept of *sublimitas*: suppose that (i.e., in reference to Fig. 17) the speed with which the projectile is thrown be that acquired by a body in free fall, which falls from *B* to the ground level *A*. Be *AE* the maximal height of the throw. The difference $BE = BA - EA$ is the *sublimitas* of the parabola.

In Proposition XXII it is required to find the entire *impetus* in a parabolic motion.

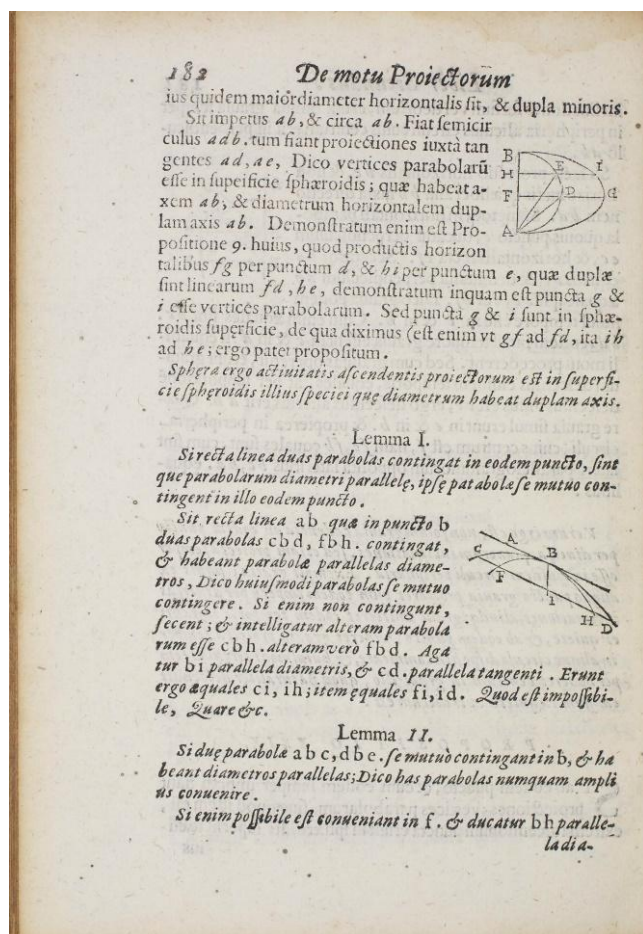
Until Proposition XXII Torricelli has sought properties which can be useful not only for mathematicians and the natural philosophers, but also for engineers who intend to construct weapons. From Proposition XXIII a more difficult and theoretically far-reaching part of Torricelli's works begins because the author discovers several geometrical properties of parabolic motion which belong to pure rational mechanics. They do not hold a pragmatic value, but a highly theoretical one.

In the first of these propositions, it is proved that all the parabolas described with the same initial *impetus* have the same *lactus rectum*. In Proposition XXV, it is required to find the focus of a parabola, given:

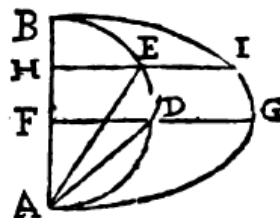
- 1) Either the *impetus* and the throw-direction;
- 2) Or the *impetus* and the range of the throw;
- 3) Or the range and the direction.

From Proposition XXIX a series of theorems begins that concerns geometrical *loci* which are quadrics.

In the XXIXth Proposition it is proved that, if from a point *A*, a projectile is thrown with the same initial speed (*impetus*), but with different directions, the vertices of all the parabolas belong to a spheroid whose major axis is double the minor (Fig. 18).



From the point *A*, the projectile is thrown. The straight lines *AD* and *AE* indicate the



different directions of the throw. The segment *AB* represents the impetus and the minor axis of the spheroid on which the vertices of the parabolas lie.

Fig. 18. The diagram used by Torricelli to prove Proposition XXIX. (Torricelli 1644, II, 2, p. 182; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 656-657) Source: Courtesy University of Oklahoma, History of Science Collections

In Proposition XXX it is proved that, if you throw a projectile from a point A , with the same speed, but with different directions, all the parabolas will be tangent to a parabolic conoid (paraboloid of revolution). The *latus rectum* of the conoid is four times the throw-height if the projectile is thrown upwards. The proof offered by Torricelli is valid for the parabolas which are the trajectories of a projectile thrown upwards. Torricelli's analysis is completed as follows: the parabola of the horizontal throw does not touch the conoid, but approaches it indefinitely (Proposition XXXI). The parabolas which are trajectories of downward throws do not touch the paraboloid. However, if you produce the parabolic trajectory from the opposite side of the throw-point, they touch.

The book, which includes 37 propositions, is closed by various theorems and problems, but what we have described is sufficient as a guide to the reader.

The *De motu gravium* has a long appendix on the motion of waters entitled *De motu aquarum*. The source of inspiration of this appendix are the works of Benedetto Castelli, defined by Torricelli as "my master"³⁷. The main result of this important writing is the so-called "Torricelli law". It can be expressed like this: given a vessel full of water, drill a very little hole in the vessel's wall so that the vertical distance between the hole and the water's surface is h , the water will fall from the hole with a speed $v = \sqrt{2gh}$. Interestingly Torricelli assumes this law as a supposition which, at the beginning of the appendix, he tries to prove experimentally. He writes:

Let us suppose that the water, which exits violently from an orifice has, in that point the same *impetus* as a heavy body, or a drop of that water which falls from the liquid's surface until the orifice³⁸.

Torricelli claims that experience confirms this principle. For, connect the hole in the vessel A (suppose it is drilled at a vertical distance h from the water's surface) by means of a little horizontal tube with another vessel B , where a small hole is drilled at the same height of the hole in A . The water will ascend from the hole in B for the height h . However, Torricelli claims that an observation is paramount: the ascent of the water in B does not reach exactly the height h . He explains the reason why this happens. He continues claiming that from each hole the water falls with a parabolic trajectory. For, the initial drops which exit the tube are as projectiles. Therefore, they describe parabolas. The following drops follow the same path. Thence, the continuous water jet has a parabolic form (Torricelli 1644, II, 2, p. 193). Once again Torricelli states that the form of the jet does not seem exactly a parabola because no experiment can take place in the vacuum.

After that, he considers a vessel whose height is AB and where holes at E , D , C are drilled (Fig. 19). He solves the problem to find the amplitude (that is the horizontal distance from the hole in the vessel and the point in which the water lands) of any parabola described by the water which exits the holes.

³⁷ The most significant work by Castelli on water's movement is *Della misura dell'acque correnti* (Castelli 1628). This is the writing which inspired Torricelli.

³⁸ "Supponimus. Aquas violenter erumpentes in ipso eruptionis punto eunde impetum habere, quem haberet grave aliquod, sive ipsius aquae gutta una, si ex suprema eisudem aquae superficie usque ad orificium eruptionis naturaliter cecidisset." (Torricelli 1644, II, 2, p. 191; Authors' translation).

maiorē partē sequentium huic instantis subiacere. Causa est impedimentum mediū, quod ad momentum corporis mobilis valde sensibilem habet rationem, multoq; maiorem quam in projectionibus quę sunt à machinis bellicis. Siquidem illic materia projectionis sunt globi plumbei, ferrei, vel saltem marmorei, hic verò linea est, & quicquid aqua. Nulli igitur mirum sit, quod cum fundamentum huius doctrine eque verū sit Thoricę loquendo, ac in projectis Galilei, practicę tamen multum ab ipsis contemplationibus aberrent experimenta, quę ad hoc ut exactiora euaderent, vel fieri deberent in medio non obstante, vel saltem gravissima materia esset adhibenda. Quamquam si quis modica altitudine, solertiq; diligentia experiri hec omnia velit, minimū quoddam, & plerumq; insensibile deesse comperiet. Experimentum, quod nobis confirmavit, has penē omnes speculationulas, factū fuit tubo quodam, imo capsula parallelepipedā, cuius altitudo passum Geometricum excedebat, cuius basis vno palmo quadrato non erat minor. Foramina verò erant rotunda, circuloque humana pupillę maiora, non perperam facta, sed solertiſſimē excavata in lamellis cupreis, tenuibus, & ad horizontem erectis. Aqua enim violenter erumpens semper directione exit perpendiculari ad illud planum ex quo crumpit, ideoq; fiebat ut emissiones nostri tubi horizontales essent.

Dato tubo ab semper pleno, & aptē perforato foraminibus d & e . hoc est quę sint figurę circularis, sitq; illorū ductus horizontalis, hoc est in tenui lamella plana pendulari. Datoq; horizonte quolibet bg , inuenire amplitudinē vniuscuiusq; parabolę. Fiat circa ab diametrū semicirculus ahb . Eritq; parabolę fluctis, ex e amplitudo dupla lineę ei quę horizontaliter ducitur in semicirculo. Et amplitudo parabolę erumpentis ex d erit dupla lineę dh . Et hoc probatur, quia cum aqua sit velut projectum quoddam, sitq; (per suppositum) ipsius punctū sublime a , erunt per Propositionem 5. Galilei, semisses amplitudinū medio loco proportiona-

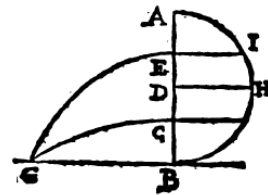
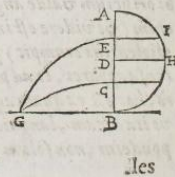


Fig 19. The diagram used by Torricelli to prove the above-mentioned theorem. In the picture the amplitude of the parabola of the water exiting from E is GB . (Torricelli 1644, II, 2, p. 194; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 680-681) Source: Courtesy University of Oklahoma, History of Science Collections.

Torricelli proves a series of beautiful theorems in which he connects his results on the motion of projectiles with the problems concerning the motion of water. For example, he is able to determine the speed with which the water exits the tube's holes, given the height of the tube.

He also poses other problems which are related to solid geometry. An interesting one, in part already considered by Castelli, concerns the forms of the solid assumed by the water exiting a hole drilled at the base of a big vessel where the water level is maintained constant (Torricelli 1644, II, 2, p. 197). Torricelli also considers other interesting situations. One, which also has a practical importance is the following: be given a tube inclined upwards in respect to the horizon with a little hole at the higher extremity. Be some water pushed in the tube from the lower extremity. It is given the point where the water jet exiting the tube lands. It is required to find the highest level reached by the jet (Torricelli 1644, II, 2, p. 198). As always in this text, Torricelli explains the best conditions to develop an experiment, which confirms the theoretical result. A further remarkable achievement concerns the quantity of water, which exits a vessel in the same time interval, but under different conditions: be given (Fig. 20) a cylindric or prismatic vessel with a little hole at the basis. Be the vessel first filled with water until the height DA (as the water comes out it must be replaced by more water so that the water level does not decrease) and afterwards until the height FH . Torricelli proves that, in the two cases, the quantity of water exiting the vessel in the time interval Δt is as the square root of DA to the square root of FH .

Posito vase *ab* siue cylindrico, siue prismatico quod in fundo perforatum sit foramine *b*. Velocitas aque exeuntis ex *b* velocitati libellæ, siue supremæ superficiæ descendents in vase, semper eadem ratione respondebit.

Quando libella aqua in vase est *ad*, fit
 velocitas *ae* quae excutitur per *b*. Tum fiat *b* ^{B²} *M*
 ut scio *ad* valis, ad sectionem orificij *b*,
 ita *ae* ad *ae*. Erunt per doctrinam Castellij, ipsa *ae*. velo-
 citas libellae *da* in decedendo. Iam circa *am* diametrum fi-
 antur *c*. & *e*. duae parabolae *me*. *me*.

Confideretur deinde alia libella *fb*. Quando *fb*. libella erit, tunc per demonstrata erit velocitas in *b* vt linea *hl*. Sed velocitas in *b* ad velocitatem libellae *fb* erit per doctrinam Castellij vt sectio *fb* ad sectionem *bc*, nempe vt *ca* ad *ae* & sic semper. Quare velocitas aquae exeuntis ad velocitatem libellae descendens in quocumq; loco confideretur, semper erit vt linea applicata in maiori parab. ad applicatam in minori; hoc est in eadem semper ratione.

Aliter etiam offenditur idem hoc modo. Intelligatur in vase ab quolibet sectio fh quæ non sit summa superficies; sit autem summa superficies d etiam l am; cum eadem quantitas æque transeat perfectionem b & per fh , erit velocius in b ad velocitatem in fh reciprocè ut sectio fh ad sectionem b ; sed sumpta sectio fh quæ uelocis est ad supremam superficiem d cum l am ponatur cylindrus, siue prismâ, ergo velocitas in b ad velocitatem supremæ superficiei d a descendentis erit uisè semper eadem ratione respondebit; nimirum semper erit ut sectio fh uasæ ad sectionem l am foris am b .

Corollarium.

Ergo quando altitudines in vase erunt $a m, h m$, erit veloci-
tas aquæ excurrentis ex b posita altitudine $a m$, ad velocitatem
excurrentis ex b posita altitudine $h m$, ut est velocitas summa su-
perfi.

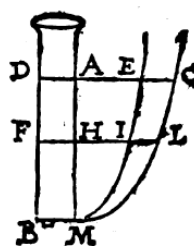


Fig. 20. The diagram used by Torricelli to prove the theorem expounded in the running text.
(Torricelli 1644, II, 2, p. 199; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 690-691)
Source: Courtesy University of Oklahoma. History of Science Collections.

Taken together, Figs. 18–20 clarify the geometrical structure underlying Torricelli’s treatment of parabolic motion.

The treatise on water is closed by two beautiful theorems. In the first one Torricelli considers a vessel whose form is a paraboloid of revolution, being the vertex the lowest point of the paraboloid. Be the vessel filled of water till a certain height. If a hole is drilled at the vertex and if, when the water exits, it is not replace by other water, the motion with which the surface of the water descends is not uniform. The second proposition generalizes and offers a precise mathematical quantification of the speed with which the water's surface in a vessel of any form descends. Thus, be given a vessel of any form (Fig. 21). The speed with which the water's surface descends in AE is to the speed in BD as

$$\frac{\sqrt{FC}}{\sqrt{GC}} = \frac{BD}{AE}.$$

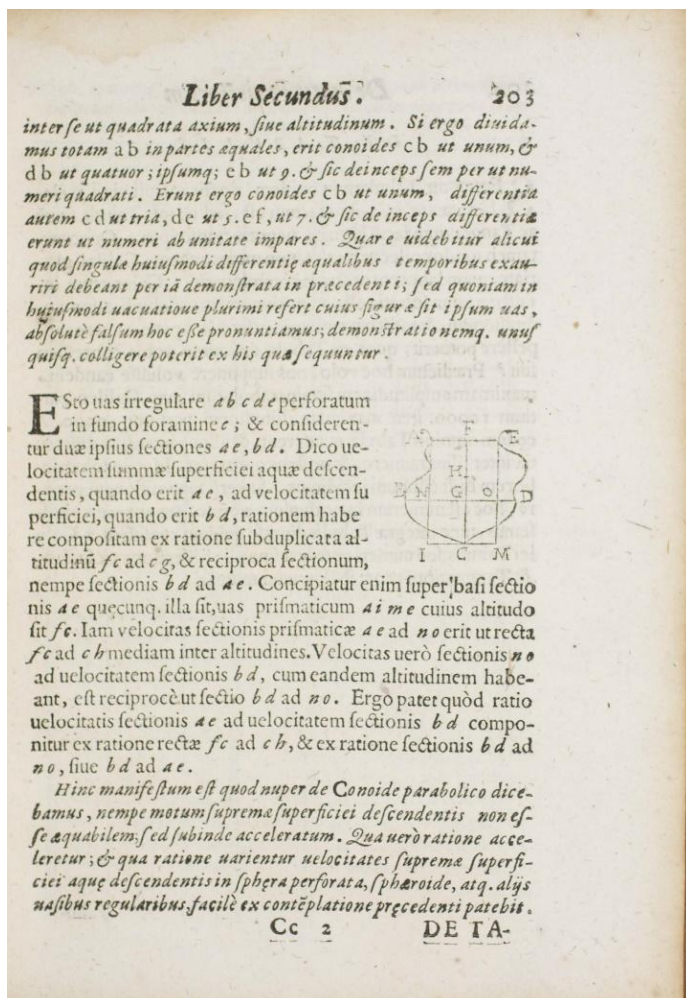


Fig. 21. The figure used by Torricelli to prove the theorem expounded in the running text.
(Torricelli 1644, II, 2, p. 203; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 698-699)
Source: Courtesy University of Oklahoma, History of Science Collections.

The results obtained in *De Motu Aquarum* can be interpreted as an appendix of Torricelli's studies on the fall of the heavy bodies because they depend on the achievements obtained in the two books *De motu gravium naturaliter descendentium*. No new principle is added. Torricelli's law, too, depends on the properties of falling bodies analysed in the two books. This notwithstanding, *De Motu Aquarum* represents a fundamental novelty in the study of the fluids' motion, where many important attainments, as we have seen, are explained. Properly speaking, Torricelli does not yet develop a theory of hydrodynamics, but his results are remarkable and represent a significant improvement in the knowledge of this discipline. From a methodological point of view, Torricelli is faithful to his geometrical approach. For, in the proofs developed in *De Motu Aquarum*, no algebraic means is used. Torricelli follows his classical style, which characterises the whole of his production. As to the epistemology of this work, the experimental aspect plays a far more important role than in the two books on the motions of the heavy bodies. Probably this depends on the fact that Torricelli's theoretical apparatus is not sufficient to prove all his results as theorems. Thus, he resorts more often than usual to the evidence of experiment. He demonstrates care in considering the conditions that might modify the experiment's results.

De Motu Gravium is closed by two further parts:

- 1) a series of tables (Torricelli 1644, pp. 204-226) where Torricelli connects the elements of the heavy bodies' motion. For example (Fig. 22), he offers the relation between the initial inclination angle of a projectile's trajectory and the amplitude of the half-parabola which is the trajectory itself. The condition is that the projectile always starts with the same speed for every

inclination angle and the maximal length of the half-parabola is posed equal to 10,000 (obtained when the inclination is 45 degrees)

*Tabula continens Altitudines semiparabolarū quarū impetus idē sit cū impetu
precedentis Tabula. Supponitur maxima altitudo part. 10000. Sunt autem
numeri Tabula semisses sinuum verorum arcuum elevationis duplorum.*

GRAD Eleuat.	ALTIT. femipar.	GRAD Eleuat.	ALTIT. Semipar.	GRAD Eleuat.	ALTIT. Semipar.
1	3	31	2653	61	7650
2	12	32	2808	62	7796
3	27	33	2966	63	7939
4	49	34	3127	64	8078
5	76	35	3290	65	8214
6	109	36	3455	66	8346
7	149	37	3622	67	8473
8	194	38	3790	68	8597
9	245	39	3960	69	8716
10	302	40	4132	70	8830
11	364	41	4304	71	8940
12	432	42	4477	72	9045
13	506	43	4651	73	9145
14	585	44	4826	74	9240
15	670	45	5000	75	9330
16	760	46	5174	76	9415
17	855	47	5349	77	9494
18	955	48	5523	78	9568
19	1060	49	5696	79	9636
20	1170	50	5868	80	9698
21	1284	51	6040	81	9755
22	1403	52	6210	82	9806
23	1527	53	6378	83	9851
24	1654	54	6545	84	9891
25	1786	55	6710	85	9924
26	1922	56	6873	86	9951
27	2061	57	7034	87	9973
28	2204	58	7192	88	9988
29	2350	59	7347	89	9997
30	2500	60	7500	90	10000

Fig. 22. The first table constructed by Torricelli.
(Torricelli 1644, II, 2, p. 207; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 706-707)
Source: Courtesy University of Oklahoma, History of Science Collections.

We will not deal any further with these tables and their explanations because they do not substantially alter the picture we are forming.

- 2) A treatise in Italian entitled “On the set square [Della Squadra]” (Torricelli 1644, pp. 226-243). This might be considered a work of practical military geometry (sometimes also applied to land surveying) in which Torricelli shows how to construct a set square which can determine the correct elevation for a projectile to reach the target. This set square is conceived to be applied to every possible machine (Cfr. Pisano and Bussotti 2014a, 2014b, 2015a). We also add that this chapter (*Liber*) is not marked as *Liber Secundum* as usually displayed in the previous pages; but it concludes all the previous chapters on motion:

End of the Books on Motion.³⁹

Therefore, as stated above, the reader should pay attention to the ordering of the page numbers of *Opera Geometrica*.

Torricelli clarifies that his treatise is addressed to the “Filosofi Geometri” (Philosopher Geometers, namely physicists and mathematicians) and not to the “Bombardieri Pratici” (practical military artilleryman, namely the military technicians, sometimes engineers, who were busy with war machines, who lacked a preliminary theoretical scientific background; See Pisano's works below; Capecchi and Pisano 2007a; Robinson 1994).

3.3 The Quadrature of the Parabola and the Cubature of the Acute Hyperbolic Solid

The final work of the *Opera geometrica* is entitled *De Dimensione Parabolae Solidique Hyperbolici Problemata duo cum Appendice De Dimensione Spatii Cycloidis et Cochleae*. With regard to the squaring of parabola, Torricelli explains that he is going to offer twenty means to solve the problem, some of them developed through the geometrical method (what we call exhaustion), some of them through the geometry of indivisibles. It is appropriate to remark that Torricelli knew (and was inspired by) Archimedes's *The quadrature of the parabola*. In this work two methods to square the parabolic segment are used: the former is mechanical because Archimedes resorted to the law of the lever and to the notion of centre of gravity. However, he did not picture the planar figures composed with an infinity of segments, which is typical of the *Method*. Obviously, Torricelli did not know the *Method*. The latter is purely geometrical.

The long *Praemium* has a remarkable methodological and epistemological value because Torricelli claims that the argument of squaring the parabola is old and the problem is completely solved. Nonetheless, a mathematical result can be important not only because it proves new theorems and solves new problems, but also because it shows the reach of a method or of a series of methods. This aspect is as important as the discovery of new theorems. Torricelli also adds a justification of his view, which is based on the utility of geometry in practical life: geometry is the most suitable discipline to sharpen ingenuity. One who is excellent in geometry will also excel in architecture, military art, arithmetical and measuring arts. These disciplines are fundamental for the health of a country both in peace and in war. Thus, he who is skilled in geometry will be also useful for the *res publica*. Since studies concerning the possibility of offering different proofs of the same theorem are among the most refined in geometry, they are among the most useful, also from a practical standpoint insofar as they help to develop geometrical skill (Torricelli 1644, III, 1, pp. 7-8). However, even if the utility of geometry for the *res publica* were not recognized, the theoretical efforts must not be despised. Otherwise, a miller should be considered more valuable than an artist. Therefore, it seems that Torricelli considered, in a sense, mathematics close to an art, but also endowed with a practical value, which the arts do not have.

The methodological importance of the *Praemium* also concerns more specific aspects of geometry: Torricelli defends mechanical methods in geometry.

³⁹ "Fine de' libri del moto" (Torricelli 1644, p. 243; Authors' translation).

At the beginning of his reasoning he refers to the ideas of those who deny the validity of such methods:

- a) The geometrical objects have no gravity, as the mechanical methods suppose;
- b) The threads through which the figures are attached at a lever are supposed to be equidistant, whereas they converge towards the Earth's centre.

Torricelli answers that, following this line of reasoning all the principles of geometry should be denied because, for example, it is false that circles have a centre, the sphere a surface and the cone a volume. It is clear that these properties are ascribed to geometrical figures by definition. Thus, it is also legitimate to assign a weight to such figures by definition. With regard to the second objection: it is true that, if considered on the Earth, the threads tend to converge towards the Earth's centre. However, if the lever is transported to the celestial regions, the threads will be parallel. Furthermore, for a geometer the fact that, in practice, the threads converge has no meaning because he can imagine (postulate) that they do not converge. From his point of view, this abstraction is perfectly legitimate. The picture, Torricelli claims, changes completely if we consider the real, physical figures. Then, it is necessary to take into account that the threads appended to a lever tend to converge, but if abstract, geometrical figures are considered, this is not a problem (Torricelli 1644, III, pp. 9-11).

Thus, from this *Praemium*, it is evident that Torricelli had an advanced image of geometry: it is an abstract discipline, based on axioms and definitions. If the axioms one introduces are not contradictory, it is legitimate to introduce and employ them. Torricelli surely has an abstract view of his discipline, not tied to anything empirical.

The Quadrature of the parabola opens with a chapter entitled *Suppositiones et definitines*. The first supposition is connected to Torricelli's principle. For, it is stated that, given any magnitude M hanging from a point which belongs to M itself, the gravity centre of M is not at rest, until it has arrived at the lowest point of its sphere (Torricelli 1644, III, pp. 11-12). This simply means that the suspension point, the gravity centre and the Earth's centre have to be collinear. Torricelli adds a brief argument which can, in fact, be interpreted as a proof. Some other very intuitive suppositions and definitions follow. There are six suppositions concerning the equilibrium of figures and two further which are not connected to the figures regarded as heavy bodies:

- 1) given a tangent t to a parabola at a point P of the parabola, if you draw from the points $Q, R, S, \dots$ of the tangent a series of straight lines parallel to the parabola's axis until they cut the parabola resp. in $Q', R', S', \dots$ it holds $QQ':RR':SS', \dots = \sqrt{PQ}:\sqrt{PR}:\sqrt{PS}, \dots$. This property, as Torricelli claims, can be deduced from Proposition XX of the first book of Apollonius' *Conics*;
- 2) be given a parabola, its axis and a chord AB perpendicular to the axis. From the points $C, D, E, \dots$ of AB draw series of straight lines parallel to the axis, which cut the parabola in $C', D', E', [\dots]$. It holds $CC':AC \cdot CB = DD':AD \cdot DB = EE':AE \cdot EB = \dots$.

After this preliminary section, the propositions begin. The first series of statements are five lemmas on the tangents to a parabola. The first three are rather clear, while it seems to us appropriate to report the fourth and the fifth ones because they connect the properties of the tangents to those of the figures considered as heavy bodies and because their proofs – especially in regard to the *fifth Lemma* are not trivial.

The fourth Lemma states that (Fig. 23) be given a parabola, a chord Ad and the tangents at A and at d . Envelop the parabola with any number of tangents drawn from points belonging to the parabolic arch Abd . The figure so generated is in equilibrium if it is suspended at the vertex B of the parabola.

20 De Dimensione Parabolæ

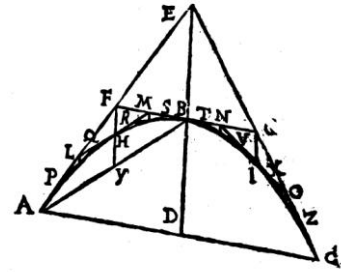
tur; & hoc fiat quotiescunq; libuerit: figura à tangentibus circumscripta, si ex vertice parabole suspendatur (posita diametro ad horizontem perpendiculari) æquiponderabit-

Est parabola abc , cuius diameter bd , & due tangentes ad ba sint ac , ce ; per verticem vero b tangens sit f bg . Deinde, demissis (ut in præcedenti Lemma) diametris fh , gi , per vertices portionum ahb , bic , tangentes ducantur lm , no . Iterumque per vertices reliquarum quatuor portionum tangentes ducantur pq , rf , tu , xz ; & sic semper donec libuerit: Dico figuram, sine potius duas figuras rectilineas à tangentibus pqr ifp , $tuxz$ g ; circumscriptas, expuncto b æquiponderare: statuta prius diametro bd ad horizontem perpendiculari.

Ponatur itaque bd diameter parabole ad horizontem perpendicularis; & rectam fg , (quamcunq; inclinationem sortiatur) concipiamus esse libram, cuius fulcrum sit in b .

Quoniam igitur applicata ab bisariam fecatur à diametro fh in puncto y ; suntq; ab , lm , parallele, erit etiam lm secta bisariam in h ; & idcirco duorum triangulorum lhm , ngo , contragavitatis sunt in fh , gi ; suntq; fh , gi ad horizontem perpendiculares, idcirco appensa centraliter erunt dicta triacula ad libram fg expunctis f & g . Acquiponderabuntque ex distantijs equalibus bf , bg . Cum ipsa quoque triacula sint equalia; nempe suboctupla eiusdem trianguli feg . Eadem prorsus ratione postea libralim, duo triacula plq , rmf appensa erunt expunctis f & m ; equi-

Draw the tangents in A and d . From the vertex draw the parallel to the basis Ad which cut the two tangents in F and G resp. From



these points draw the parallel to the axis which cut the parabola in R and in I . Draw the tangents LM and NO in these two points. From L, M, N, O , draw the parallels to the axis and the tangents at the points where such parallels cut the parabola and so on. The figure, or rather as Torricelli claims, the two rectilinear figures $PQRSTP$ and $TUXZGT$ are in equilibrium if they are suspended in B

Fig 23. The figure used by Torricelli to prove Lemma IV.

(Torricelli 1644, III, 1, p. 20; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 820-821)

Source: Courtesy University of Oklahoma, History of Science Collections.

The fifth Lemma considers a chord AC of a parabola whose vertex is B . Be given the tangent c at C and b at B and the parallel a to the axis at A . Be $F = c \cap b$ and $G = a \cap b$. Cut FG in E , so that $FE = 2EG$. The figure $ABCFG$ is in equilibrium if suspended in E (Fig. 24). This Lemma is proved by a very ingenious reasoning based on the exhaustion method.

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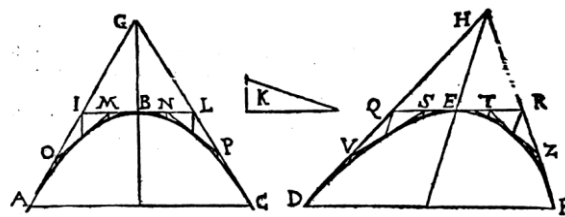
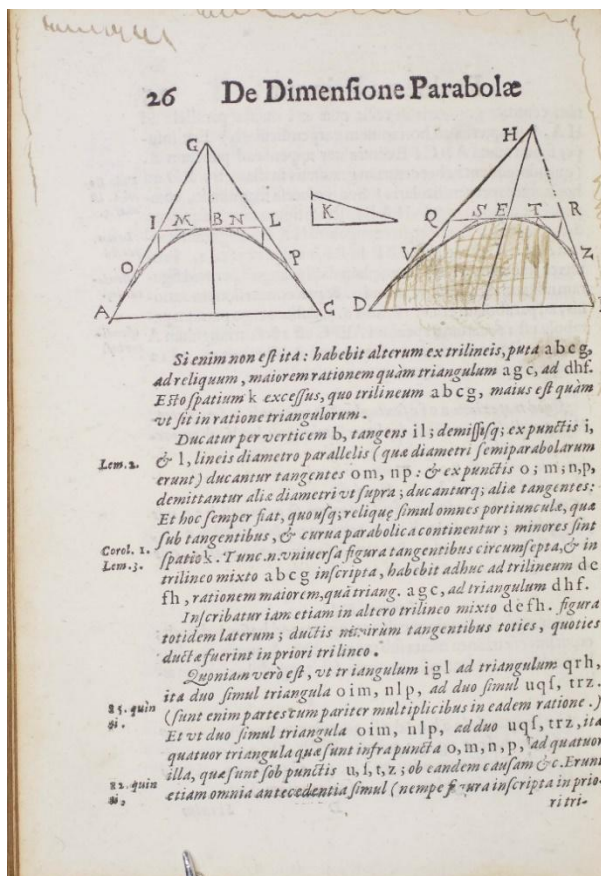


Fig. 25. The diagram used by Torricelli to prove Lemma VI.

(Torricelli 1644, III, p. 26; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 832-833)

Source: Courtesy University of Oklahoma, History of Science Collections.

It must be noted that in this case the law of the lever equilibrium is not used. In the Proposition II Torricelli offers the second demonstration on the area of the parabolic segment. In this case the proof, based on Lemma VI, does not entail mechanical elements. Thus, it is in the style of the second proof given by Archimedes in *The quadrature of the parabola*. Torricelli's work continues with a series of four fascinating Lemmas, numbered VII, VIII, IX, X, the most interesting of which is the last one, proved by a long and admirable geometrical reasoning, where exhaustion plays a fundamental role. In Lemma X, Torricelli proves that:

Lemma X. If from a parabola, half of the inscribed triangle is removed, the whole remaining mixtilinear figure is in equilibrium around the base's point of the remaining triangle which divides such a base so that its part whose extremity is on the curve is four times the part whose extremity is on the diameter.⁴⁰ (See also Fig. 26).

⁴⁰ "Lemma X. Si ex parabola auferatur dimidium trianguli inscripti, tota reliqua figura mixta aequiponderabit ex puncto basis reliqui trianguli, in quo sic ea dividitur, ut pars ad curvam terminata quadrupla sit illius, quae terminator a diametro." (Torricelli 1644, III, p. 30; Authors' translation; Torricelli's italic style).

30 De Dimensione Parabolæ

dentur inscripta, detracto prius triangulo abc , equiponderare ex puncto d . Item reliquum parabola, dempto triangulo abc , equiponderare ex d .

Lemma X.

Si ex parabola auferatur dimidium trianguli inscripti, tota reliqua figura mixta æquiponderabit ex puncto basis reliqui trianguli, in quo sic ea diuiditur, ut pars ad curuam terminata quadrupla sit illius, quæ terminatur ad diametrum;

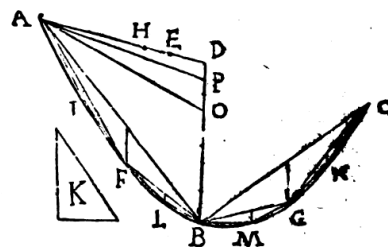
Esto parabola a bc inuersa; eiusq; diameter bd ita sita utatur ut ad horizon-tem sit perpendicu-laris; Detractoq; semitriangulo in-scripto dbc ; see-atur ad basis reliqui semitrianguli, in- quinq; partes equa-les; quarum una sit de . Dico huiusmodi figuram ex puncto c suspensam, equiponderare.

Nisi enim equiponderet; Cum recta ad sit libra, cuius fulcrum est in c , & magnitudo a $fbgc$, constans ex duabus portionibus parabolicis, appensa sit ad punctum d secundum centrum graui-tatis ipsius; Reliquum autem triangulum abd altera magnitu-do appensa sit ad punctum h (sumpta dh tertia parte totius da); Altera ex his duabus magnitudinib. preponderare necesse erit.

Ponamus primo preponderare duas portiones a fb , b gc ; & sit excessus quo preponderant, equalis spatio K .

Inscribatur euidenter intra duas portiones parabolicas figura multilatera, ita ut omnia simul segmenta parabolica circumeli-cta minora sint spatio K . Tunc enim preponderabit adhuc figu-ra inscripta multilatera a ifl bm g nc ba .

Acci-



Explanation of Torricelli's statement: Be ABC the parabola, BD a diameter perpendicular to the horizon. Remove the triangle DBC . In the diagram this operation is already developed. Divide the remaining segment AD of the basis into five equal parts, so that $AE=4ED$.

Fig. 26. The diagram used by Torricelli to prove Lemma X

(Torricelli 1644, III, p. 30; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 840-841)

Source: Courtesy University of Oklahoma, History of Science Collections.

The further series of lemmas allowed Torricelli to reach, in Proposition III, another demonstration of the parabolic segment's area. Torricelli's work proceeds in the same manner we have explained. That is, a lemma or a series of lemmas on the equilibrium of particular figures connected to the parabolic segment is proved and, after any single series of lemmas, a new demonstration concerning the area of the parabolic segment is given. The exhaustion method is used to prove Lemma XI, which is exploited in Proposition IV, where two different manners to square the parabola on the basis of this lemma are offered. In Proposition V a further proof by exhaustion is given where there is no reference to mechanical means. Thus, this proof is in the style of Archimedes's second demonstration offered in *The quadrature of the parabola*. Lemma XII, which is used in Proposition VI, is proved by exhaustion, but it is not connected with mechanical properties. Thence, also the proof expounded in Proposition VI is not mechanical. Lemma XIII concerns the equilibrium of certain figures and is used in Proposition VII to offer a further quadrature. In this Lemma, the proof of which is very long, the exhaustion method is used. The proof explained in Proposition VIII is based on the series of lemmas XIV-XVII. They are very interesting. The first one concerns solid geometry: consider two cones ABC and DEF . Divide the sides of the triangles through the axes of the two cones in the same number of equal parts. Through the division points draw planes parallel to the bases of the cones. Each cone will be divided into a cone and a series of truncated cones. In each truncated cone inscribe a cylinder. All the cylinders in each cone will have the same height. The lemma claims that the volume of the cone ABC is to that of DEF as all the cylinders in ABC are to the cylinders in DEF . The proof is based on a direct method. Lemma XV, which is not proved by exhaustion, is an interesting proposition of infinitesimal character: be given a mixtilinear triangle included between a parabola, its tangent and a

straight line parallel to the diameter. In the triangle, it is possible to inscribe a figure formed of parallelograms of equal height. This figure is less than the triangle by any given quantity (Fig. 27).

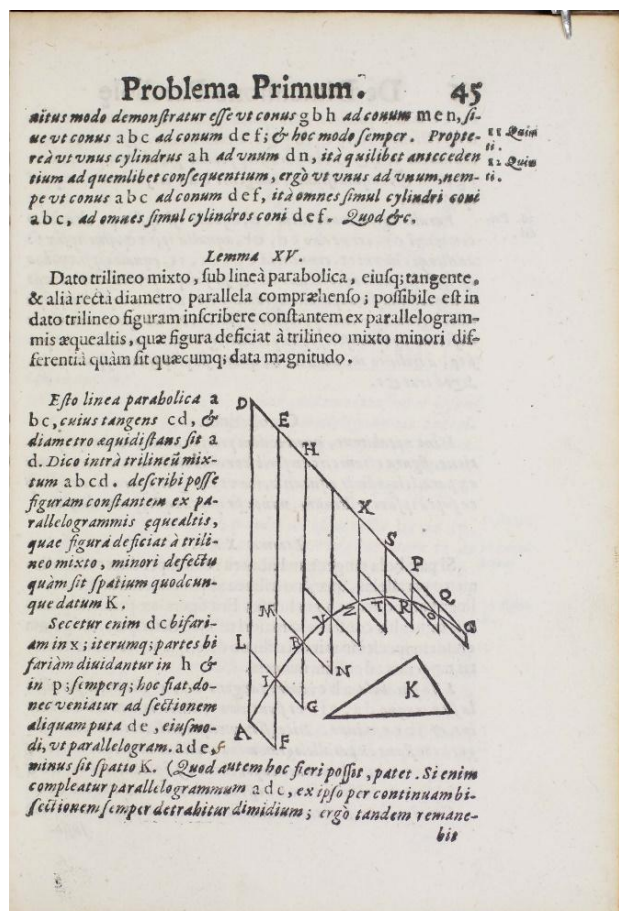
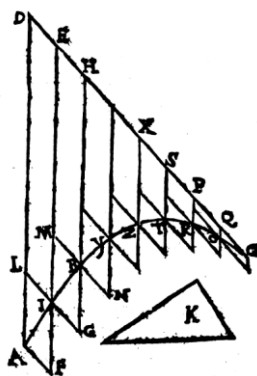


Fig. 27. The diagram used by Torricelli to prove Lemma XV.

(Torricelli 1644, III, 45; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 870-871)

Source: Courtesy University of Oklahoma, History of Science Collections.

AO is the parabolic segment, OD is the tangent and AD the straight line parallel to the axis. The mixtilinear triangle is AOD . The figure formed by parallelograms is constructed like this: divide the tangent segment OD into two equal parts in X . Bisect each of the two parts resp. in H and in P and so on. Continue to bisect until reaching the segment DE which is the side of the parallelogram $DAFE$ whose area is less than the given quantity K . Torricelli constructs a series of parallelograms $CO, OR, TR, \dots$, whose sum, according to Torricelli's constructive criteria is equal to the parallelogram $DAFE$. This is the inscribed figure with the required features. Note that this construction has several similarities with Newton's lemmas on integral calculus in the *Principia*. (Newton 1687, I, Section I, Lemmas II, II, IV, pp. 28-30; Cfr. Pisano and Bussotti 2025).



A further Lemma in which the exhaustion method is used is the 17th one. Torricelli proves that, given the parabolic segment ABC of base AC , its tangent CD and two straight lines BE and AD parallel to the parabola's axis, the area of the mixtilinear triangle $ABCD$ is to the area of the mixtilinear triangle CBE as the cube of DC is to the cube of CE . In this case, there is no reference to the mechanical method. Proposition VIII offers Torricelli's 9th proof by exhaustion of the quadrature of parabola. Proposition IX is introduced by two lemmas, numbered XVIII and XIX. The former might be called "arithmetical". It concerns the mutual relations among four series of magnitudes. The style of this Lemma is similar to that of Archimedes's arithmetical lemmas (see the first two propositions of *Conoids and spheroids*; the propositions 10 and 11 of the *Spirals*; Proposition 23 of *The quadrature of parabola*). Note that no algebraic notation is used (Cfr. Pisano and Bussotti 2015c). The demonstration is written in words, without any symbolization. This lemma is important because it is also used in the second part of *De mensura parabolae*, where the proofs through indivisibles are expounded. The Lemma states: be given four series of magnitudes. The first a and the third c be constant magnitudes. The second b_i and the fourth d_i variable, but under the condition that, for every pair b_j and d_j , it holds $a : b_j = c : d_j$, then it holds $\sum a : \sum b_j = \sum c : \sum d_j$.

The Lemma XIX states that, given a parabolic segment of basis AB , through the extreme A draw the tangent at the parabola and through B the parallel to the parabola's axis. Be C their intersection point. The area of the triangle ABC is three times the area of the parabolic segment. The proof is given through a reasoning by exhaustion without resorting to considerations on the lever. Thence, the purists too should judge this proof geometrical. The demonstration is long and needs many subtleties, of which Torricelli was capable. As a matter of fact, from this Lemma the proof concerning the area of

the parabolic segment follows as a clear corollary, which is shown in Proposition IX. The first part of the book on the quadrature of the parabola is closed by Proposition X, which is a piece of geometrical virtuosity. The arithmetical Lemma XVIII is used as well as Lemma XV. Since no mechanical element is employed, this proof can, to all effects and purposes, be considered geometrical. In what follows, we present a table where we summarize which, among the eleven exhaustion quadratures proposed by Torricelli, are purely geometrical and which resort to mechanical instruments. In the first column the number of propositions is reported, in the second one the lemmas used to prove it, in the third one the mechanical or geometrical character of the proof. We also indicate where mechanical reasoning occurs. It is, however, appropriate to remember that, according to Torricelli, there is no substantial difference between a purely geometrical proof and a mechanical one. However, according to other mathematicians, the first of whom was Archimedes himself, such a difference exists. In the following, we present (Table 3) Propositions upon Lemmas and kinds of Proofs.

For ease of comparison, the different geometrical procedures are listed in Tab. 3.

Tab. 3 Propositions, Lemmas and Kinds of Proofs (Torricelli 1644, III)

Number of Proposition	Lemmas used	Kind of proof
Proposition I	Lemmas I-V	Mechanical Lever used in Lemma V
Proposition II	Lemma VI	Geometrical
Proposition III	Lemmas VII-X	Mechanical Lever used in Lemma X
Proposition IV Two proofs are completed	Lemma XI	Mechanical Lever is used in Lemma XI and in a further unnumbered lemma used in the second proof, which is mechanical, as well.
Proposition V		Geometrical
Proposition VI	Lemma XII	Geometrical
Proposition VII	Lemma XIII	Mechanical Lever is used in Lemma XIII
Proposition VIII	Lemmas XIV-XVII	Geometrical
Proposition IX	Lemmas XVIII and XIX	Geometrical
Proposition X	Lemmas XV and XVIII	Geometrical

Before analysing the section in which Torricelli used the indivisibles method to square the parabola, an observation seems appropriate to us on the links between the exhaustion method and the infinitesimal, and, in particular, integral calculus. It is clear that exhaustion implies that the concept of limit is known. In some propositions this is particularly conspicuous. Torricelli's Lemma XV is an example, given its similarity as far as the geometrical construction is concerned to Newton's lemmas on the integral calculus expounded in the *Principia*. Indeed, Torricelli is looking for two quantities whose difference becomes arbitrarily small when another quantity, the side of a parallelogram, becomes arbitrarily small. The fact that the concept of function did not yet exist is not a problem in this respect. The most skilled geometers knew the notion of limit intuitively and knew how to use it without running into errors. However, there is a substantial difference between exhaustion method and integral calculus: the former is associated neither with a calculus nor with an algebraic symbolization of the geometrical quantities. This is a crucial aspect. There is no standard way to develop a reasoning by exhaustion. An invention is necessary in every single demonstration, though the general scheme of the proof is invariable. The need to use the *reductio ad absurdum* is the counterpart of the direct calculation. In integral calculus, you have a series of standard procedures, which allows a direct calculation of the required areas and volumes. Therefore, our position is in the middle between that of the scholars who see profound similarities between exhaustion method and integral calculus and those who consider them to be two completely different mathematical procedures.⁴¹

⁴¹ The relation between exhaustion method and integral calculus has been debated for a long time in the literature on Archimedes. This Introduction is not the appropriate place to debate such a literature. Broadly speaking, it is possible to state that there are two tendencies. The first one underlines the similarities between the two methods, whereas the other highlights their irreducible differences. Without any claim to be exhaustive, we

The brief introductory remarks to the second part of Torricelli's work on the square of the parabola are significant: first, Torricelli claims that the quadrature of the parabolic segment can be deduced from some of the most famous theorems proved by the ancients:

- 1) The cylinder's volume is three times the inscribed cone's volume;
- 2) The cylinder's volume is $\frac{4}{3}$ the volume of the inscribed sphere;
- 3) The cone's gravity centre G is on the axis so that the distance between G and the cone's vertex is three times the distance between G and the cone's base;
- 4) After the first rotation the spiral's area is $\frac{1}{3}$ of the first circle's area. Reciprocally, assuming the square of parabola, the four theorems follow. How are these deductions possible?

Torricelli's answer is that they are possible thanks to indivisibles. Torricelli's opinion on Cavalieri's geometry of indivisibles is clear: such geometry is admirable. It is not a new discovery: probably it is the heuristic through which the ancients arrived at their discoveries. Torricelli's words are so clear that no interpretation is necessary:

I would not dare geometry of indivisibles to be a new invention. Rather, I incline to think that the ancient geometers resorted to this method to discover the most difficult theorems. After that, in the proof they preferred a different method both to hide the secrets of their art and to avoid offering to envious detractors any occasion for criticism. It is anyway certain that this geometry permits an admirable energy saving in the discovery of new truths and allows us to establish new and almost inscrutable theorems. This cannot be done through the ancients' methods.⁴²

These are really prophetic words if we think of Heiberg's discovery of Archimedes's *Method*! But they are not only prophetic, they imply a legitimation of Cavalieri's method against the detractors, such as Guldin. Torricelli found no logical problem in indivisibles. They work, they allow us to solve difficult mathematical problems; thus, they are legitimate. We know that Cavalieri himself was, instead, very concerned about the logical validity of his indivisibles; he tried to justify their use, and also attempted to identify a correct and an incorrect conceptualization by distinguishing between *retto transito* and *obliquo transito* (Cfr. Andersen 1985; Giusti 1980). Torricelli's attitude was more pragmatic. He explained the principles in a series of lemmas (Torricelli 1644, III, p. 57) on Cavalieri's geometry, but, with the term *principles* he meant the *basic mathematical statements* rather than the *logical bases* (Pisano 2010a; and Capecchi and Pisano 2007ab).⁴³

This section of Torricelli's work opens with the first demonstration of the parabola's quadrature given by the technique of indivisibles (Proposition XI). Let us see Torricelli's ways of reasoning (Fig. 28).

Be ABC the parabolic segment, CD the tangent at an extreme of its base and AD the parallel to the diameter drawn from the other extreme. Complete the parallelogram AE . Describe a circle around AD , regarded as the base both of the cone whose vertex is C and of the cylinder $ADEC$. Draw FG parallel to AD . It is an indivisible of the parallelogram $ADEC$ and IB is an indivisible of the mixtilinear triangle $ABDC$.

mention: a) Bachmakova 1963 (The author offers a modernizing treatment of several of Archimedes' results), b) Archimedes 1897; Heath 2002 (see Introduction), c) Marzbach-Boyer 2011 (see chapter 6), d) Rufini 1926 (it is dedicated to Archimedes' mechanical method, but there are many considerations on exhaustion), e) Zeuthen 1896, pp. 177-187, f) Saito 2013. On our side, as clarified in the running text, we incline toward an historical intermediate position. More, see recently: Bussotti 2019; Pisano and Bussotti 20225

⁴² "Quoad autem haec Indivisibilium Geometria noivum penitus inventum sit, equidem non ausim affirmare. Crediderim potius veteres Geometras hac methodo usos in inventione Theorematum difficillimorum, quamquam in demonstrationibus aliam viam magis probaverint, sive ad occultandum artis arcanum, sive ne ulla invidis detractoribus proferretur occasio contradicendi. Quicquid est, certum est hanc Geometriam mirum esse pro inventione compendium, et innumera quasi imperscrutabilia Theoremata, brevibus, directis, affirmativisque; demonstrationibus confirmare; quod per doctrinam antiquorum fieri minime potest" (Torricelli 1644, III, p. 56; Authors' translation).

⁴³ Cfr. This does not mean that Torricelli was not interested in establishing the conditions under which the technique of indivisibles is correctly applicable. As we will see, he also offers a series of examples in which the indivisibles are applied incorrectly.

Through AG , draw a plane parallel to the circle AD . It holds:

$$FG:IB = DA:IB = DC^2:IC^2 = DA^2:IG^2$$

This proportion holds for any other straight line parallel to DA . For example if we draw $F'G'$ and indicate the points where it cuts resp. the tangent and the parabola by I' and B' , it is:

$$F'G':I'B' = DA:I'B' = DC^2:I'C^2 = DA^2:I'G'^2$$

Because of Lemma XVIII, all the first magnitudes, namely the parallelogram regarded as composed of the indivisibles $DA, FG, F'G', \dots$ are to all the second magnitudes, that is $DA, IB, I'B'$ regarded as composing the mixtilinear triangle $ABCD$, as all the third magnitudes, namely all the circles of diameter DA , which are the indivisibles of the cylinder, are to all the circles of variable diameter $I^{(i)}G^{(i)}$ which are the indivisibles of the cone. Therefore, the parallelogram is three times the mixtilinear triangle. Hence, the parabolic segment is $1/3$ of the triangle ADC , which is 4 times the triangle whose basis is AC and whose height is that of the segment. Thus, the theorem is proved.

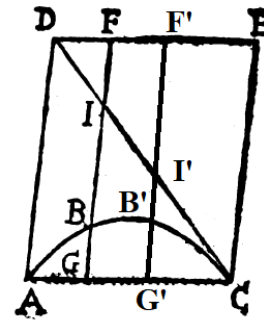
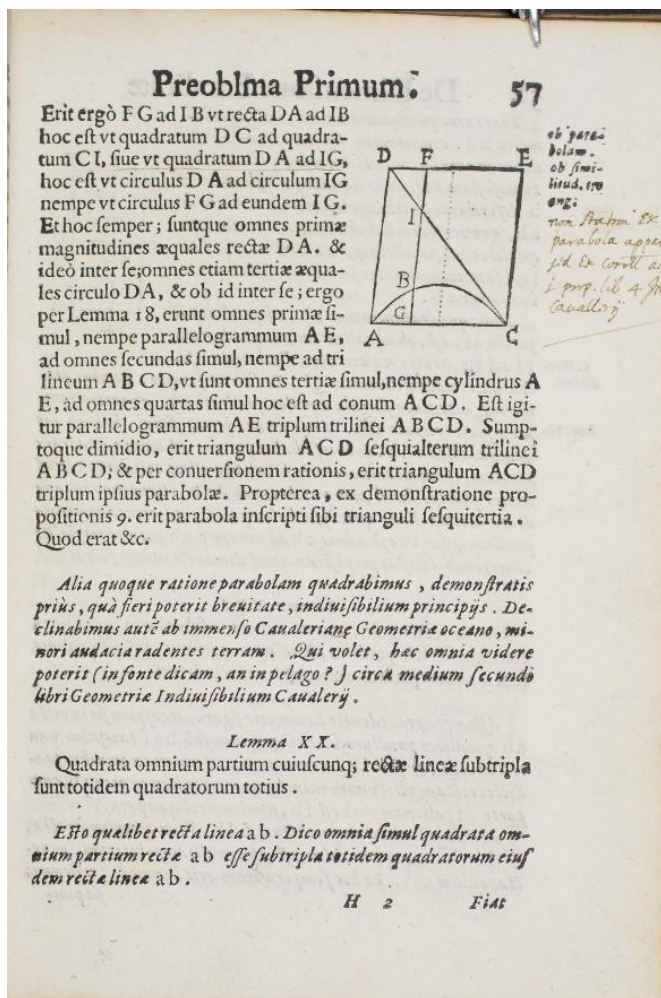


Fig. 28. Torricelli's diagram in which the segment $F'I'B'G'$ has been added to make Torricelli's proof clearer (Torricelli 1644, III, p. 57; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 898-899)

Source: Courtesy University of Oklahoma, History of Science Collections.

This is the first way in which Torricelli squared the parabola relying upon indivisibles. No lemma was then added for this first demonstration. However, to obtain the other proofs, several further lemmas were conceived and proved by Torricelli.

The indivisibles method is applied both in the proofs of the lemmas and of the propositions. The proof of Proposition XII is based on two lemmas, the 20th and the 21st. The former is a very audacious statement because Torricelli considers a segment AB and the series of its subsegments, for example, those whose extreme is A . Let us denote them by AX . Torricelli believes that the concept “all the subsegments AX ” is perfectly meaningful. Now to each subsegment associate the whole segment AB . Torricelli proved that the sum of all the squares of the subsegments AX is $1/3$ the sum of the squares of all the equinumerous segments AB . He, thus, believed in the existence of the set of all the infinite subsegments AX , which is actually infinite. The idea that a continuum of geometrical dimension n is composed of an infinite set of continua of geometrical dimension $n-1$ is typical of the indivisibles method. However, here Torricelli is imagining a different situation because he is not regarding the segment AB as composed of its infinite points. Rather he is considering such a segment as the support of an actual infinity of subsegments whose variable length is AX . They do not compose the segment AB . This is a very interesting aspect of Torricelli’s view.

Lemma XXI is another statement concerning indivisibles. Armed with these two lemmas and with Lemma XVIII, which is used in many proofs included in this second part of his quadrature of the parabola, in Proposition XII Torricelli offers the second demonstration based on indivisibles.

In Proposition XIII, only the XVIIIth Lemma is used. The proof is based on Archimedes’s theorem that the cylinder’s volume is $4/3$ the volume of the inscribed sphere.

The following proposition in which Torricelli dealt with the quadrature of the parabola is interesting because it is based both on the indivisibles method and on the principle of the lever. Therefore, it is a mechanical quadrature through indivisibles. For, Lemma XXII proves that, given a lever L , suspend a series of magnitudes at n different points of the lever (the magnitudes are, in general, different). Suspend a second series of magnitudes at the same points. Be the magnitudes of the two series proportional. A single point of the lever will be the equilibrium centre of both the series of magnitudes. The second Lemma on which the proof of Proposition XIV relies is a geometrical statement based on the lever’s equilibrium. This time the indivisibles are also used and the similarity between the proof of Torricelli’s Lemma XXIII and those given by Archimedes in his *Mechanical Method* is impressive. Let us see: given a parabolic segment of base AC , draw the tangent in C and a parallel to the axis in A . Be D their intersection point. The mixtilinear triangle $ABCD$ is in equilibrium if it is suspended in a point H of the tangent such that $CH=3HD$ (Fig. 29).

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erit ut ghi simul ad l. Ergo due magnitudines. cde, & f. habebunt idem punctum æquilibrium quod habent due magnitudines ghi & l. Et sic etiam si sint plures magnitudines, usque in infinitum, quod erat propositum &c.

Lemma XXIII.

Si parabola tangentem habuerit ad basim, ex altera verò parte lineam diametro parallelam. Trilineum comprehensum sub curuâ parabolâ, sub tangente, & sub parallela prædictâ, æquipo-
nderabit ex puncto tangentis ubi ea sic diuiditur, vt pars ad contactum terminata reliquæ sit tripla.

Est parabola abc, cuius tangens ad basim sit cd; equidistans diametro sit ad. Dico trilineum mixtum abcd æquipo-
ndere ex puncto tangentis cd, ubi ea sic diuiditur ut pars versus contactum c, reliqua sit tripla.

Concipiatur figura ita ut da ad horizon-
tem sit perpendicularis; & circa diametrum da intelligatur circulus, qui sit basis con-
uerticem habentis in puncto c.

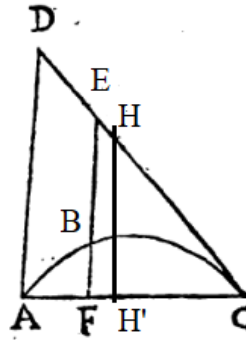
Sumpto iam quolibet puncto e ducatur
ef equidistans ipsi da; & per ipsam transeat planum paralle-
lum basi coni.

ob para-
bolam. — Erit ergo recta da ad eb, ut quadratum dc ad ce; sine
ut quadratum da ad ef, hoc est ut circulus da ad ef. Et
hoc semper, ubicumq; sit punctum e. Ergo cum ad libram d c
pendant ab istis punctis magnitudines duorum ordinum pro-
portionales ut in præcedenti lemmate imperatum est, habebunt
omnes magnitudines simul primi ordinis (hoc est omnes lineæ
trilinei abcd, siue ipsum trilineum) idem punctum æquilibrii,
quod habent omnes magnitudines simul secundi ordinis (hoc est
omnes circuli coni acd, sine idem conus). Conus autem æqui-
ponderat ex puncto quod secatur cd ita ut pars ad c reliquæ sit
tripla

Proof of Lemma XXIII

Around DA as diameter draw a circle regarded as the base of a cone whose vertex is C . Take any point E on the tangent. Draw EF parallel to DA and through EF a plane parallel to that of the circle DA . It is

$$DA:EB = DC^2:CE^2.$$



Thus, according to Lemma XXII, all the magnitudes of the first series, that is the segments composing the mixtilinear triangle $ABCD$, have the same equilibrium point as the magnitudes of the second series, which compose the cone. Since the cone is in equilibrium if it is suspended at the point H of DC such that $CH=3HD$, the same holds for the trilinear.

Fig. 29. Adapted. Diagram for Proving Lemma XXIII.

For convenience of the reader, we have added the straight line HH' (see right side of the table) which does not exist in Torricelli's diagram. (Torricelli 1644, III, p. 62; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 908-909) Source: Courtesy University of Oklahoma, History of Science Collections.

Through these two lemmas, in Proposition XIV Torricelli is able to square the parabola in two different manners, both based on the indivisibles method enriched by that of the lever's equilibrium.

The following Proposition XV reaches the quadrature of the parabola in two different ways. This time the technique associated by Torricelli with the indivisibles is no longer that of the lever's equilibrium, but that of the infinite series of proportional quantities. The lemmas on which the two proofs are based are numbered XXIV-XXVII. The most interesting among them is the XXVIth one because Torricelli asks to find a segment whose length is equal to that of a segment composed of an infinite number of decreasing segments. The Lemma states: given infinite straight lines (segments) in continued proportion, making up a decreasing progression, find the straight line equal to the sum of all the infinite given straight lines. In Torricelli's time, it was well known that a sum of infinite decreasing addends can produce a finite value. For example, this was known for the geometrical series whose general term is less than one⁴⁴. However, here it seems that Torricelli contemplates the possibility of developing a calculation with infinities, which is remarkable. Obviously, this calculation is subject to the usual limitation which connotes Torricelli's mathematics: no algebraic symbolism is used.

Lemma XXVIII is noteworthy from a conceptual point of view. It states that given two infinite series of segments $AB, CD, EF, \dots$ and $BG, DH, FI, \dots$, each of them in continued geometrical proportion such that $AB:BG = CD:DH = EF:FI: \dots$, the sum of all the segments belonging to the first

⁴⁴ Cfr. *Tractatus de latitudinibus formarum* (1361; Oresme 1486) by Nicholas Oresme. In the eight question of the third book, entitled *De mensura et intensione in infinitum quarundam difformitatum* ("The measure and intension to infinity of some differences") he proves that the altitude of a finite surface can be arbitrarily increased, so that its form is changed, but not its surface. Oresme proved through a graphic method that the geometrical series of general term $\frac{1}{2}$ converges.

series is to the sum of all those belonging to the second series as $AB:BG$. Torricelli claims that this lemma is included in Euclid's *Elements* (Euclid 2007, V, 12)⁴⁵. However, he offered a different demonstration because almost all the mathematicians think that Euclid was referring to a finite number of magnitudes, while he devised a proof completely applicable to the case of an infinite number, too. This is a further indication that he considers it legitimate to develop a calculus with infinite quantities. This is not exactly equivalent to claiming that Torricelli is on the road which led to infinitesimal calculus because:

- 1) He used no algebraic symbolization;
- 2) The relation between actual and potential infinite does not seem completely clear. For, while claiming that he considers the sum of an infinite number of magnitudes, Torricelli seems to refer to something as the sum of all the magnitude of the series, whose number is actually infinite, rather than to the sum of an arbitrary number of such magnitudes (a potential conception of infinity). The infinitesimal calculus refers to such sums in the second meaning. Torricelli's ideas in this Lemma seems almost closer to the calculus of infinite series envisioned by Bolzano in his *Paradoxien des Unendlichen* (Bolzano 1851, §§ 29-33, pp. 43-55).

In Proposition XVI Torricelli develops an interesting technique: he conceives an infinite series of triangles, included in the parabola, whose areas decrease in continued proportion and an infinite series of triangles out of the parabola whose areas also decrease in an analogous continued proportion (Fig. 30). The sum of the triangles of the first series generates the parabola, the sum of the triangles of the second series generates the mixtilinear triangle whose two sides are the two concurrent tangents at the extremes of the parabolic segments' base and whose third side is the parabolic arch. Since the first triangle of the first series is double the first triangle of the second series, through a relatively clear reasoning, it is possible to obtain the parabola's quadrature.

⁴⁵ "If there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following." (Euclid 2007, p. 142).

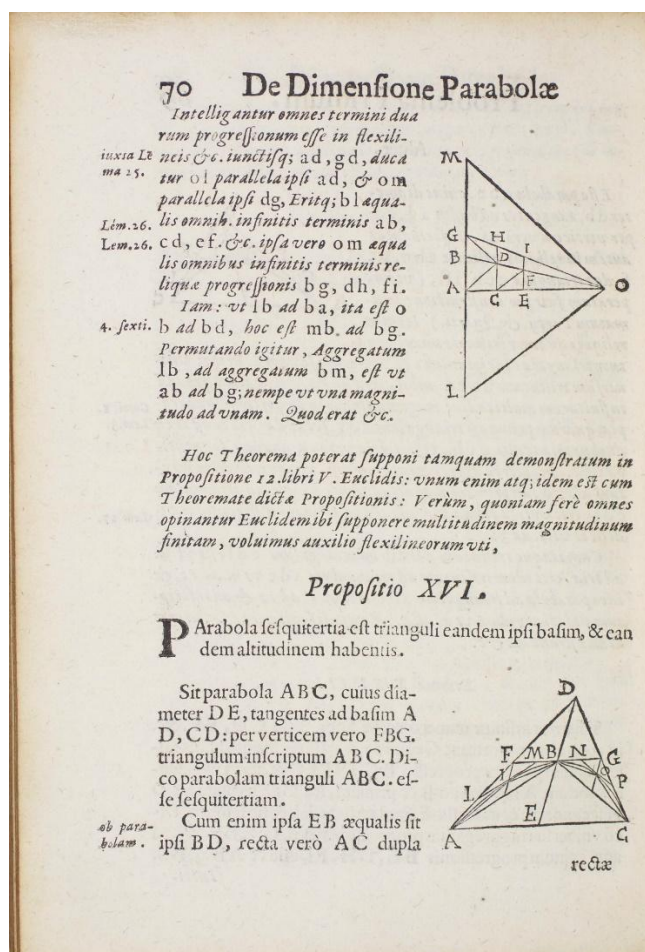


Fig. 30. The diagram used by Torricelli to prove Proposition XVI. The series of the triangles internal to the parabolic segments are ABC, AIB, COB , etc. The series of the external triangles is DFG, LMF, PNG , etc. (Torricelli 1644, III, p. 70; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 924-925)
 Source: Courtesy University of Oklahoma, History of Science Collections.

In Proposition XVII the parabola is squared through the indivisibles technique by resorting the property proved by Archimedes in his *Spirals* (Proposition XXIV) according to which the area of the first circle is three times the area of the spiral. Torricelli's proof also exploits a further statement on the series of proportional magnitudes (Lemma XXIX).

Propositions XVIII and XIX show two further methods to square the parabola which are based on mechanic considerations. Torricelli assumes as a premise Archimedes's seventh proposition proved in the second book of *On the equilibrium of planes*: the gravity centres of two similar parabolic segments divide the diameters in the same proportion. The Lemma which is premised, the XXXth one, shows that given a parabolic segment, its gravity centre divides its axis so that the part between the gravity centre and the vertex is $\frac{4}{3}$ the other part. It is here that Archimedes's proposition is exploited.

The further method to square the parabola (Torricelli 1644, III, Proposition XIX, p. 78) through indivisibles is important to understand the role played by infinitesimal entities within Torricelli's geometry. This also includes the relationship between geometry–infinitesimals because it is based on a Lemma previously considered by Torricelli (Torricelli 1644, III, Proposition XVIII, Lemma XXXI, p. 76 [pp. 75–78]). At the end of the Proposition XVIII, and before Lemma XXXI, he claimed:

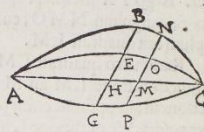
Author, Most illustrious. Antonio Rocca eminent Geometer⁴⁶

⁴⁶ "Authore Io. Antonio Roccha praestandi Geometra praestans Geometra." (Torricelli 1644, III, Proposition XVIII, p. 76; Authors' translation).

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ro subduplam, & æqualiter inclinam.

Est parabola abc , cuius diameter bh ; sintq. due alie parabole aec , agc . in eadem basi. Diametri vero he , hg , utraq. subdupla sit diametri hb : sed æqualiter ad basim inclinata. Dico parabolam abc æqualem esse figuræ $aecg$.

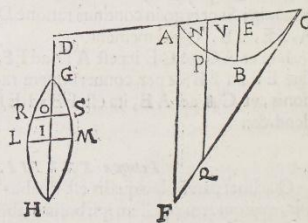


Sumatur enim quodlibet punctum in basi ac ; & sit m ; ductaq. pmn æquidistante ad diametrum bh . Erat bh ad nm , ut rectangulum ahc , ad rectangulum amc ; siue ut recta he ad mo . Et permutando ut bh ad he , ita erit nm ad mo . Quare nm dupla erit ipsius mo . Eodem penitus modo ostendetur nm dupla etiam ipsius mp , Ergo tota nm æqualis est ipsi op . Et hoc semper. Propterea omnes simul lineę figurę abc , (nempe ipsa parabola abc) æquales erunt omnibus simul lineis figurę $aecg$, (nempe duabus parabolis aec , agc .) Quod erat &c.

Propositio XX.

Parabola sesquitercia est trianguli eandem ipsi basim, & eandem altitudinem habentis.

Est parabola A BC , cuius diameter BE concipiatur ad horizontem perpendicularis, & ipsa parabola inuersa statuatur. Producat CA in D , ita ut æquales sint CA , AD ; & sit D C lib.



ra.

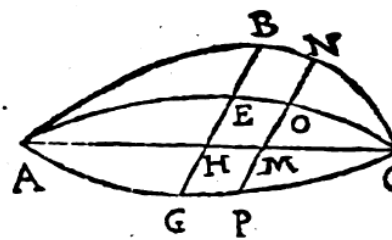


Fig. 32. The diagram used by Torricelli to prove the Lemma XXXIII.

(Torricelli 1644, III, p. 82; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 944-945)

Source: Courtesy University of Oklahoma, History of Science Collections.

The quadrature of the parabola in Proposition XX is interesting because Torricelli produces the base AC of the parabola by a segment $AD=AC$ and regards AD as a lever whose fulcrum is A . Suspend the parabola in its position and the figure $GLHM$, composed of two parabolas as in lemma XXXIII, at the extreme D of the lever (Fig. 33). Starting from this construction, Torricelli is able to develop a beautiful reasoning *à la* Archimedes (we mean the mechanical method) which allows him to square the parabola.

On this basis, we now turn to a detailed discussion of the proofs presented by Torricelli.

There are several causes for interest underlying *De dimensione parabolae*:

- 1) Mathematics is not only a discipline whose aim is to obtain new theorems and solve new problems. The methodological aspect is also important. Different demonstrations of the same proposition open different conceptual perspectives. For this reason, they are significant.
- 2) The method of exhaustion and the method of indivisibles both involve infinity. The former with infinity in its potential form. The concept of limit exists, but there is no technique concerning calculation with limits, neither a general procedure to obtain quadrature and cubatures in a standard manner as will be the case with integral calculus. This is the reason why it is necessary to resort to the *ab absurdum* reasoning. On the contrary, in the indivisibles technique, the concept of actual infinity seems involved. In a sense, this admirable technique seems a calculation with actual infinity. This aspect is particularly conspicuous in Torricelli, who extends such conception of infinity to the sum of series, as well.
- 3) *De dimensione parabolae* is admirable from a conceptual point of view, but it is once again appropriate to point out that Torricelli lacked an arithmetical–algebraic technique, which is also typical of the other great Italian mathematicians, such as Cavalieri. Of course, this was an enormous disadvantage compared to the European mathematicians who used an algebraic symbolism. In Cavalieri’s and Torricelli’s epoch this gap was not yet completely tangible because of their ability and because algebraic technique applied – to summarize – to infinitesimal problems such as those of tangents, maxima, minima, quadratures and cubatures was still in the initial phase. However, in the following decades the gap became evident and Italian mathematics lost its importance within the evolution of European science.

This work by Torricelli concludes with two appendices dedicated to the cycloid. The former is very important because Torricelli proves that the area of the cycloid is three times the circle’s area. He demonstrates this statement in three different manners: the first and the third through indivisibles, the second by exhaustion. From a historical point of view, it must be recalled that Torricelli had a priority dispute with Roberval as to the quadrature of cycloid. Torricelli, who had always been inclined to avoid any polemic, this time complained about Roberval’s behaviour because, after having received a letter from Torricelli with the determination of the cycloid’s gravity centre and the quadrature, Roberval claimed to have already reached the same results, though in inverse order.

Torricelli opens his Appendix writing that no mathematician had squared the cycloid. He attempts to offer a justification of their failure: if you try to determine, as evidently the mathematicians of the early 17th century did, the area of the cycloid experimentally, the experiment does not seem to offer precise indications. Thus, such mathematicians arrived at thinking that the ratio between the area of the generating circle and that of the cycloid could be irrational. These words by Torricelli probably means that they first tried to determine the cycloid’s area experimentally. One method might be to suspend a tiny circle and a tiny cycloid generated by that circle and of its same material at the extremes of a lever and to determine the point of equilibrium. Obviously, these empirical methods can be useful to have an idea of the result, but are useless to gain an accurate result.

In what follows we will offer the three proofs given by Torricelli and compare them with the one provided by Roberval. This is useful to have a clear idea of the ability with which Torricelli used different mathematical methods and to show that he and Roberval resorted to such different procedures to square the cycloid (although both employed indivisibles) that plagiarism must be excluded.

The analysis that follows focuses on the different procedures adopted by Torricelli and Roberval, with the aim of clarifying their methodological independence.

The first proof works like this (Fig. 34):

qualis sit periphæria circuli genitoris ac. Quod quidem non adeo obscurum est: Nam tota periphæria ac se ipsam in conversione commensuravit super manente recta ab.

Quæritur nunc quam proportionem habeat spatium cycloïdale adb ad circum suum genitorem ac? Oſtendimusque, Deo dante, triplum esse. Demonstrationes tres erunt, inter se penitus diversæ. Prima, & tertia per novam Indivisibilem Geometriam nobis amicissimam procedent: secunda verò per duplicem positionem, more veterum recepto; ut utriusque fautoribus satisfiat. Ceterum, hoc moneo; principia fere omnia, quibus aliquid per Indivisibilem Geometriam demonstratur, ad solitam antiquorum demonstrationem indirectam reduci posse: quod à nobis factum est, ut in multis alijs, ita etiam in primo, & tertio sequentium Theorematibus; sed ne lectoris patientia nimium adhuc abuteremur plura omittenda censuimus, tresque tantum demonstrationes exhibemus.

THEOREMA I.

Omne spatium quod sub linea Cycloïde, & recta eius basi continetur, triplum est circuli sui genitoris; siue sesquialterum trianguli eandem basim, & eandem altitudinem habentis.

Esto Cycloïdis linea abc descripta à puncto c circuli cd ef dum ipse circumvertitur super manente basi af. (consideramus autem semicycloïdem, & semicirculum tantum ad evitandam figure confusionem.) Dico spatium abc f triplum esse semicirculi cd ef; siue sesquialterum trianguli acf.

Accipiantur duo puncta h, & i in diametro cf. æquè remota à centro g. Ductisq. hb, il em æquidistanter ipsi fa, transiant per puncta b, & l semicirculi obp, mln, æquales ipsi cdf, & contingentes basim in punctis pn.

Nanife-

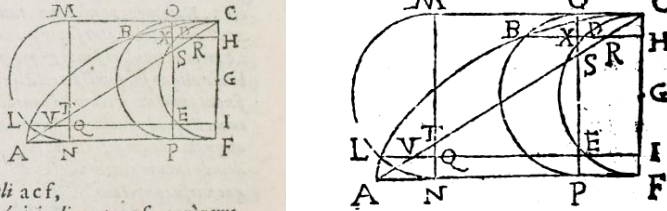


Fig. 34. The diagram used by Torricelli to square the cycloid.

(Torricelli 1644, III, p. 86; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 952-953)

Source: Courtesy University of Oklahoma, History of Science Collections.

The demonstration is developed on the half cycloid $ABCF$, while $CDEF$ is half the generator circle. On its diameter CF , take H and I so that $HG = GI$. Therefore, $CH = IF$. Draw CM, HB, IL parallel to AF . Through the points B and L where these parallels cut the cycloids, draw the half circles OBP, MLN , which touch the cycloid's base in P and N resp. It is $XD = IE = XB = QL$ and arc $OB = \text{arc } LN$. Furthermore, $CR = VA$. Given the definition of cycloid, the semi-circumference MLN is equal to AF , arc $LN = AN$, arc $LM = NF$ and arc $BO = PF$. Thus, $AN = PF$, consequently $AT = SC$ and, hence, $VT = SR$ and $VQ = XR$. Therefore, it is

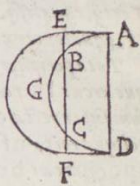
$$LV + BR = LQ + BX = EI + DH$$

This is always true wherever you take the points I and H on CF provided that they are equidistant from the centre G . All the segments of the figure $ALBCA$ are, thence, equal to all the segments of the semi-circle $CEDF$. The area of the triangle ACF is double the area of the semi-circle. Therefore, by composing, the area of the half-cycloid is $3/2$ the area the triangle ACF and, hence, three times that of the semi-circle. This is the first published determination of the cycloid's area.

Let us now analyse the second proof which is based on exhaustion. Two lemmas are necessary. The first one (Fig. 35) claims that:

Demonstratur; quoniam cum sint aequales semicirc. dempto communi segmento b g c, additisque communibus trilincis e b a, c f d. clarum erit propositum.

Quando vero detur casus quod segmentum nullum sit, tunc brevior, faciliusq. demonstratio erit. Facile etiam per eandem prosthapheresim ostenditur arcuatam sectam à linea ipsi f d parallela equale esse rectangulo aequalto, & super eadem basi constituto.



Given the rectangle $AEFB$, let the two circles with diameters AD and EF be constructed. The mystilinear figure $AEIFDG$ has the same area as the rectangle. Torricelli named *arcuatam* such a figure. The proof is very trivial.

Fig 35. The diagram used by Torricelli to prove Lemma I (particular).

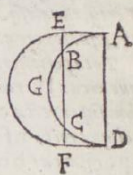
(Torricelli, 1644, III, p. 88; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 956-957)

Source: Courtesy University of Oklahoma, History of Science Collections.

Lemma II is fundamental and is proved by exhaustion (see Fig. 36).

Demonstratur; quoniam cum sint aequales semicirc. dempto communi segmento b g c, additisque communibus trilineis e b a, c f d, clarum erit propositum.

Quando vero detur casus quod segmentum nullum sit, tunc brevior, faciliusq. demonstratio erit. Facile etiam per eandem prosthapheresim ostenditur arcuatam sectam à linea ipsi f d parallela aequale esse rectangulo aequalto, & super eadem basi constituto.



Lemma I. I.

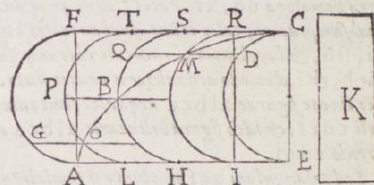
Esse lineam cycloidalem a b c descriptam à puncto c semicirculi c d e dum constructur super manente a c. Compleatur rectangulum a f c e,

fiatq. circa diametrum a f semicirculus a g f. Dico cycloidem a b c secare bifariam arcuatam a g f c d e.

Si enim ita non est, erit utiq. alterum ex duobus trilineis f g a b c, a b c d e, magis quam dimidium eiusdem arcuati. Esto & ponatur alterum ex istis (quodcumq. sit) puta a b c d e maius quam dimidium arcuati. Sitq. excessus, quo trilineum superat semissem arcuati, aequalis spatio enidam K.

Secetur bifariam a c in h; & iterum h e in i: & sic fiat semper donec rectangulum aliquod i e c minus reperiatur spatio K. Tunc dividatur integra a c in particulas aequales ipsi i e, & per puncta divisionum l, h, i, transeant semicirculi aequales ipsi c d e semicirculo, tangentes basim in punctis l, h, i, secantesq. cycloidem in o, b, m, per quae puncta agantur recta g o, p b, q m d aequidistantes basi a c.

Eri



Lemma II states what follows:

Let the half cycloid ABC be considered as well as the rectangle $AFCE$. The cycloid bisects the area of the arcuatam $AGFRDEIA$.

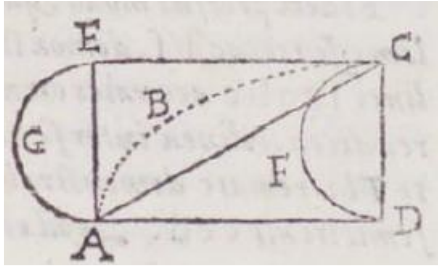
Proof. The exhaustion is used. Torricelli argues as runs: suppose the Lemma be false, then one of the two trilineals $FGABC$ and $ABCDE$ (suppose the latter) would be greater than half the arcuatam. Let us pose equal to K the difference between the area of the trilineal and half the arcuatam. Bisect AE in H , HE in I , and so on until arriving at the rectangle $LRCE$ smaller than K . Now divide the segment AE in parts equal to IE . From the division points L, H, I draw the perpendiculars LT, HS and IR (only this one is drawn in the figure) to FC and construct with diameters LT, HS and IR the equal half circles indicated in the figure. They cut the cycloid in the points O, B, M respectively. Through these points be drawn the straight lines GO, PB, QMD parallel to AE and FC . By construction, the arcuatam OH is equal to the arcuatam GL , the arcuatam BI is equal to the arcuatam PH , the arcuatam ME equal to the arcuatam QI . This means, Torricelli claims, that the whole figure composed of arcuata and inscribed in the trilineal is equivalent to the circumscribed figure minus the arcuatam $IMRCDE$. This arcuatam is equivalent to the rectangle RE , which is smaller than K . Therefore, the figure inscribed in the trilineal will be greater than half the arcuatam $AFDGE$, and, hence, greater than the trilineal $FGABC$. But it is equal to the other figure composed of arcuata and described in the trilineal $FGABC$. This is an important step of the proof depending on the nature of cycloid. For, the arc OL is equal to the segment LA because of the definition of cycloid, namely equal to the segment IE and to the arc RM . Hence, the arcuatam is equal to the arcuatam MC , and so on for all the other arcuata. Therefore, this inscribed figure should be greater of the trilineal in which it is inscribed, but this is absurd. The supposition that that the trilineal $FGABC$ be bigger than half the arcuatam $AGFCDE$ is falsified in the same manner. Thence, the lemma is true.

Fig 36. The diagram used by Torricelli to prove Lemma II.

(Torricelli, 1644, III, p. 88; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 956-957)

Source: Courtesy University of Oklahoma, History of Science Collections.

Now it is easy to prove that the area of cycloid is three times that of the generating circle. For (Fig. 37), let ABC be the cycloid generated by the circle CFD . Complete the rectangle $CDAE$ and draw the half-circle AGE . The area of the triangle ADC is twice that of the half-circle CFD . For, since ABC is a cycloid, the half-circumference CFD is as long as the segment AD . Furthermore, CD is equal to the diameter, and this proves what stated. This means that the rectangle $CDAE$ is four times the half-circle CFD . Thus, the arcuatam $AGECFD$ (Lemma I) is four times the same half-circle. This implies (Lemma II) that the trilineal $ABCD$ is twice the half-circle. Ergo, the space $ABCD$, that is the area of the half-cycloid, is three times the half-circle CFD , as it was necessary to prove.

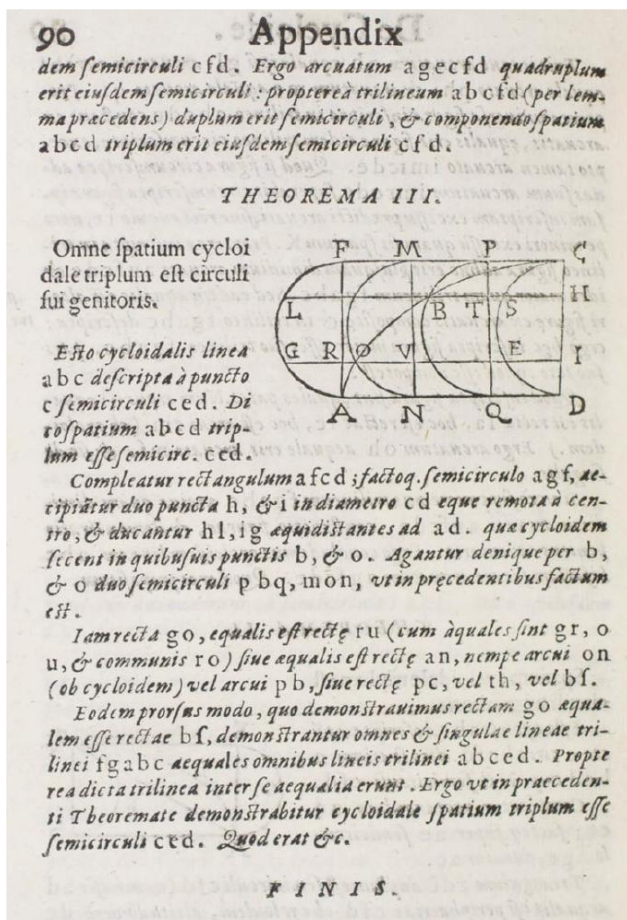


The diagram used by Torricelli to prove that the area of cycloid is three times that of the generating circle.

Fig 37. The diagram used by Torricelli to prove that the area of cycloid is three times that of the generating circle (particular). (Torricelli, 1644, III, p. 89; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 958-959) Source: Courtesy University of Oklahoma, History of Science Collections.

In this demonstration the exhaustion method is used to prove the fundamental Lemma II.

Let us now move to analyse the third and last demonstration given by Torricelli regarding the cycloid's area (Fig. 38). It is carried out by indivisibles. Torricelli provides a further example of his mathematical virtuosism in using such a technique. For this last proof is extraordinarily brief and direct.



Let ABC be the cycloid described by the half-circle CED . Draw the triangle $AFCD$. Let the points H and I be equidistant from the centre of the half-circle. Draw HL and IG parallel to AD and FC . These segments cut the cycloid in B and O respectively. Through B and O draw the two half-circles PBQ and MON . The segment GO is equal to RV and to AN , namely to the arc ON , because of the definition of cycloid, to the arc PB , to the segments PC , TH and BS . Here the technique of indivisibles enters the scene. For, Torricelli claims, as GO is equal to BS , all the single lines of the trilineal $FGABC$ are equal to all the single lines of the trilineal $ABCDE$. This means that these trilineals have the same surface. Therefore, to the same reasoning used in the previous proof, the area of cycloid will be equal to three times that of the generating circle.

Fig. 38. The third proof given by Torricelli.

(Torricelli 1644, p. 90; Pisano-Dhombres-Radelet de Grave-Bussotti, 2024, pp. 960-961).

Source: Courtesy University of Oklahoma, History of Science Collections.

It is remarkable that the fundamental Lemma II is proved in the second demonstration by resorting to exhaustion, while in the third proof by using indivisibles. For, claiming, as in the third proof, that the trilineals $FGABC$ and $ABCDE$ have the same area is exactly the same as claiming that the cycloid bisects the area of the arcuaturn $AGFCDA$.

For a comparison with Torricelli's demonstrations, let now analyse the one given by Roberval. In 1634 Roberval squared the cycloid proving its area by three times that of the generating circle.

However, his proof was kept secret for ten years, until 1644, when Mersenne published it in his work *Cogitata physico-mathematica*. In 1693 the Académie Royale des Sciences published a collective book with several works of scholars who had been members. Large part of such publication features Roberval's works⁴⁸.

In the first proposition of *De trochoide et ejusque spatio*⁴⁹ Roberval proves what follows (Fig. 39):

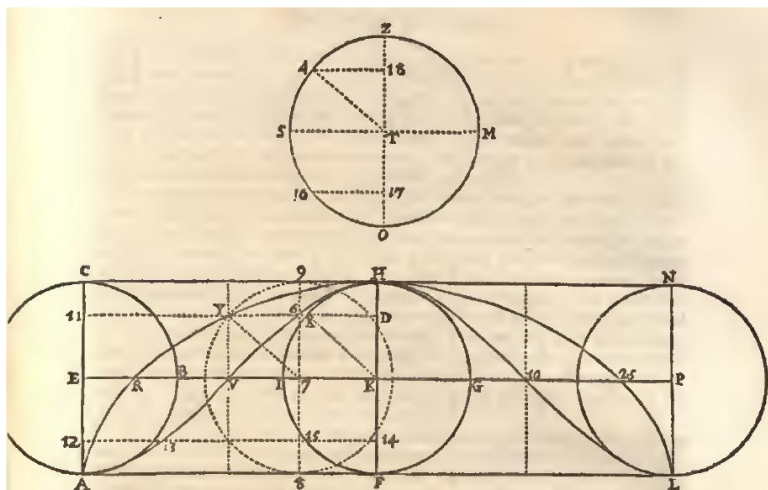


Fig 39. The diagram used by Roberval to prove his Proposition I (particular) (Roberval, 1693, p. 249).

Proposition I: Consider a point as Y belonging to the cycloid when the rotation of the generating circle is smaller than 180 degrees. Be HF the position of the diameter of the generating circle when its rotation is 180 degrees. Draw YD perpendicular to HF . The segment YD will have a portion external to the circle $HIFG$. This portion YX is equal to XH , where X is the intersection between YD and the circle $HIFG$.

The lengthy and verbose proof is developed through elementary geometry without resorting either to exhaustion or to indivisibles. This proposition has five corollaries. The fifth one is important because Roberval introduces a curve that will have a crucial role in squaring cycloid: point 6 (Fig. 39) is the intersection between YD and the diameter of the generating circle in a given position. While the centre of this circle moves on the straight line EK until reaching K , the instantaneous position of the point 6 is given by points such as $A, V, 6, H$, so describing a curve. After one complete rotation the point describing the cycloid and the one generating this new curve coincide once again in L . Roberval called the curve $AV6H10L$ *comitem* or *sociam* (Roberval 1693, p. 252) of cycloid. The usual English translation is “companion curve”⁵⁰. The cycloid and its companion curve coincide in A, H, L .

The second proposition states what follows (Fig. 39, *ibid.*, p. 252): from any point of the cycloid, such as Y , consider any straight line perpendicular to HF . In the specific case it will cut the companion curve in 6, the generating circle in X and its diameter in D . Roberval claims that $XY=6D$. Analogously, $RI=VK$, and so on. The proof is easy, so that we refer directly to Roberval text. This proposition has five easily provable corollaries. The second one is important because it is used in the proof concerning the area of cycloid. It states that $Y6=XD$.

The third proposition (*ibid.*, pp. 253-254) is as follows: from any point of the cycloid, draw two any straight lines perpendicular to AC and LN and symmetric with respect to EK (Fig. 39). The segment joining one point of the companion curve intercepted by one of the two symmetric straight lines with its intersection with CA is equal to the segment belonging to the symmetric straight line which connects the point where such a straight line cuts the companion curve and the point where it cuts HF . For example, the segment 13-14 is equal to the segment 6-11 and 13-12 is equal to 6- D . Though being

⁴⁸ In our context we are not interested in following the events connected with Roberval's squaring of cycloid and with its priority dispute with Torricelli, that we mention at the end of this section. Our aim is only to compare the mathematical methods used by them. For the entire story we refer to Dhombres 2024. Roberval's work on cycloid was published in 1693 jointly with other his contributions. We refer to this edition.

⁴⁹ (Roberval 1693, pp. 246-277. First Proposition pp. 248-250). Roberval uses the name trochoid to indicate the cycloid.

⁵⁰ It is not difficult to find the parametric equation of the companion curve: if a is the radius of the generating circle and the rotation angle t is assumed as parameter, such an equation is $\begin{cases} x = at \\ y = a(1 - \cos t) \end{cases}$. However, it plays no role in Roberval's reasoning.

the proof a little bit more difficult than that of the previous proposition, it relies neither on exhaustion nor on indivisibles, thence we refer directly to Roberval's text.

The fourth proposition derives from the third one (*ibid.*, p. 254). For it states that the companion curve bisects the surface of the rectangle $CAFC$, restricting to a rotation of 180 degrees of the generating circle (Fig. 39). Here the technique of indivisibles is applied because, according to the third proposition all the segments composing the surface $AVHF$ are equal to all those composing the surface $ACHV$, as it is conspicuous. Therefore, by the principle of indivisibles, the two surfaces are equal.

The fifth proposition (*ibid.*, pp. 254-255) is a trivial consequence of the previous one because it states – using a very complicate and verbose language – that, if we indicate by R the radius of the generating circle, the area subtended to the companion curve is $2\pi R^2$.

The sixth proposition is a fundamental step for squaring the cycloid. It states that the space $ARHVA$ included between the cycloid and the companion curve is equivalent to half the circle generating the cycloid (*ibid.*, p. 255). The proof is developed through indivisibles (Fig. 39). Roberval consider the segment YD . Because of the second corollary of the second proposition $Y6=XD$. This is also the case for all the other straight lines drawn parallel to the basis AL . For example, $RV=IK$. Therefore, because of the principle of indivisibles the area of $ARHVA$ is the same as the area of the half-circle FIH .

This proposition has three corollaries. In the first one Roberval claims that, through an argument like that expounded, it is possible to prove that the figure $ARYHIFA$ is equivalent to $AVHFA$ (*ibid.*, p. 256). This is proved by indivisibles because, according to the second proposition, $YX=6D$ (Fig. 39) and this is the case for the analogous segments intercepted by every parallel to AL . \

It is easy to prove the second corollary (*ibid.*, p. 256) which states that the two figures $ARHCA$ and $AVHIFA$ (Fig. 39) are equivalent. The proof of the third corollary is equally easy. It is demonstrated that the following four surfaces are equivalent: $ARHCA$, $ARHVA$, $AVHIFA$, and FIH .

Finally, Roberval demonstrates the seventh proposition and its corollary (*ibid.*, p. 256). The statement of the proposition is easier explained as follows rather than literally following Roberval's enunciation: the area of cycloid is equivalent to the area of the companion curve more the area of the generating circle. This depends on the fact, proved in the sixth proposition, that the figure $ARYH25L10HVA$ is equivalent to the generating circle. Taking into account proposition 5, where it is proved that the area of the companion curve is twice that of the generating circle, it follows, as a corollary, that the area of the cycloid is three times that of the generating circle.

The proof by Roberval and the third one given by Torricelli have some similarities insofar as both resort to an auxiliary figure: Roberval to the companion curve of the cycloid and Torricelli to the arcuatum. For the rest, the two demonstrations, although based on indivisibles, rely upon different lemmas and have different structures, as the reader can understand reading our presentation. This shows the richness and malleability of the method of indivisibles which, hence, around the half of the 17th century seems a promising way to tackle the problems connected with areas and volumes of curvilinear or mystilinear figures as well as to the computation of barycentres. After a few decades it was overcome by infinitesimal calculus, but it was, anyway, a powerful method. \

Finally, also without analysing other aspects of the question, the mathematical difference between Roberval's and Torricelli's demonstrations and the fact the latter gave three proofs for the area of cycloid, excludes any possible question of plagiarism.⁵¹

Now we will analyse Torricelli's result which was probably the most surprising for the whole European mathematical milieu: the volume of the hyperbolic acute solid.

The essay *De solido acuto hyperbolico* is very rich from a methodological, conceptual and mathematical point of view.

Let us summarize the main reasons of interest behind this work:

- 1) Torricelli shows the existence of a solid whose extension is infinite, but whose volume is finite. He was perfectly aware of the novelty of his result. For, he writes:

So far, in the Schools of Mathematics you find measures of figures which are limited on all sides. Among all the solids, of which the ancient and modern authors – with many difficulties – determined the measure, no one has an infinite extension, as far as I know. If you consider an

⁵¹ We avoid the details of the two other proofs given by Torricelli, neither will we consider the scholium postponed by Torricelli to the Appendix (Torricelli 1644, III, Appendix De dimensione cycloidis pp. 85–90).

infinitely extended solid or a flat figure, everyone thinks the size of this figure to be infinite. In fact, a solid of infinite length exists, which is so tiny that, however produced to infinity, does not exceed the size of a small cylinder. It is the solid generated by the hyperbola, to which this treatise is dedicated. So far, no one considered this solid, but it gives rise to a plurality of interesting theorems. If my judgement were not veiled by passion, I would say that in the whole history of geometry none of the figures considered so far is more interesting.⁵²

Thus, from a mathematical standpoint the examination of an infinitely extended solid having a finite volume is a remarkable novelty.

- 2) With regard to the methods used to prove his theorem, Torricelli offered two demonstrations: one based on exhaustion and the other one on indivisibles. However, here Torricelli clearly explains his predilection for indivisibles: the exhaustion method is perfect from a geometrical point of view, but it is not heuristic and, relying only on it, geometry had made little progress. The true scientific method is that of indivisibles. Torricelli is entirely explicit:

As to the proof method, we will demonstrate a single remarkable theorem in two manners: with the indivisibles and following the ancients' method. Actually, my result was discovered through the geometry of indivisibles, which is the real scientific and, so to say, natural proof-method.⁵³

Torricelli continues, claiming that the ancients' geometry was extremely limited because they did not know or did not use indivisibles, while the admirable Cavalieri opened the door to calculate areas and volumes of numerous and variegated classes of solids.

- 3) To calculate the volume of the acute hyperbolic solid, Torricelli did not resort to the usual Cavalieri's indivisibles. One might call them "rectilinear". Rather, he invented and used the "curved indivisible". The idea is that a figure can be interpreted not only as an aggregate of rectilinear segments, but also of curvilinear ones. Torricelli pointed out that his idea was approved by Raffaello Magiotti. To explain his new method, Torricelli proposed fourteen examples of propositions, which were known, but were proved without using curved indivisibles. He specified when there was a concordance between his result and one of those obtained by Archimedes. We present two of his examples, the first and easiest one, relative to a flat figure and the 12th which, instead, concerns a solid figure.

Taken together, the analysis of Figs. 34–39 makes clear that the procedures adopted by Torricelli differ in a fundamental way from those employed by Roberval, thus establishing their methodological independence and ruling out any hypothesis of plagiarism.

⁵² "Hactenus enim in Mathematicis Scholis repertae sunt dimensiones figurarum ab omni parte finem habentium; quandoquidem inter omnia solida, quae ab antiquis et modernis Auctoribus multiplici conatu ad mensuram redacta sunt, nullum adhuc, quod ego sciam, ullam dimensionem habuit extensione infinitam. Imo statim atque proponatur sive solidum aliquod, sive figura plana, cuius aliqua extensio in infinitam distantiam procedat, unusquisque cogitabit huiusmodi figuram infinite magnitudinis esse debere. Attamen solidum habet Geometria, longitudine quidem infinitum, sed tanta praeditum subtilitate, ut licet in infinitum producat, exigui tamen cylindri molem non excedat. Tale erit solidum illud ab hyperbola genitum, quod huius libelli contemplatione prosequemur; intactum hucusque ab aliis, et multiplici curiosaque Theorematum varietate faecundissimum; eo usque ut, nisi me fallat affectus, universa Geometria inter hactenus consideratas figuras nullam habeat curiositate abundationem." (Torricelli 1644, IV, p. 93; Authors' translation).

⁵³ "Quo ad methodum demonstrandi, unicum quidem, et praecipuum Theorema duplici conatu ostendemus, et per indivisibilia, et more Veterum. Quamquam (ut vera fateamur) primo inventum sit per Indivisibilium Geometriam; qui sane verus est demonstrandi modus scientificus, semper directus, et ipsi naturae germanus." (Torricelli 1644, IV, p. 94; Authors' translation).

The first example chosen by Torricelli regards the circle's area (Fig. 40). Archimedes in the first Proposition of *De mensura circuli* proved that the area of the circle BD is equal to that of the right triangle of which one small side BC is equal to the rectified circumference and the other small side AB equal to the radius.

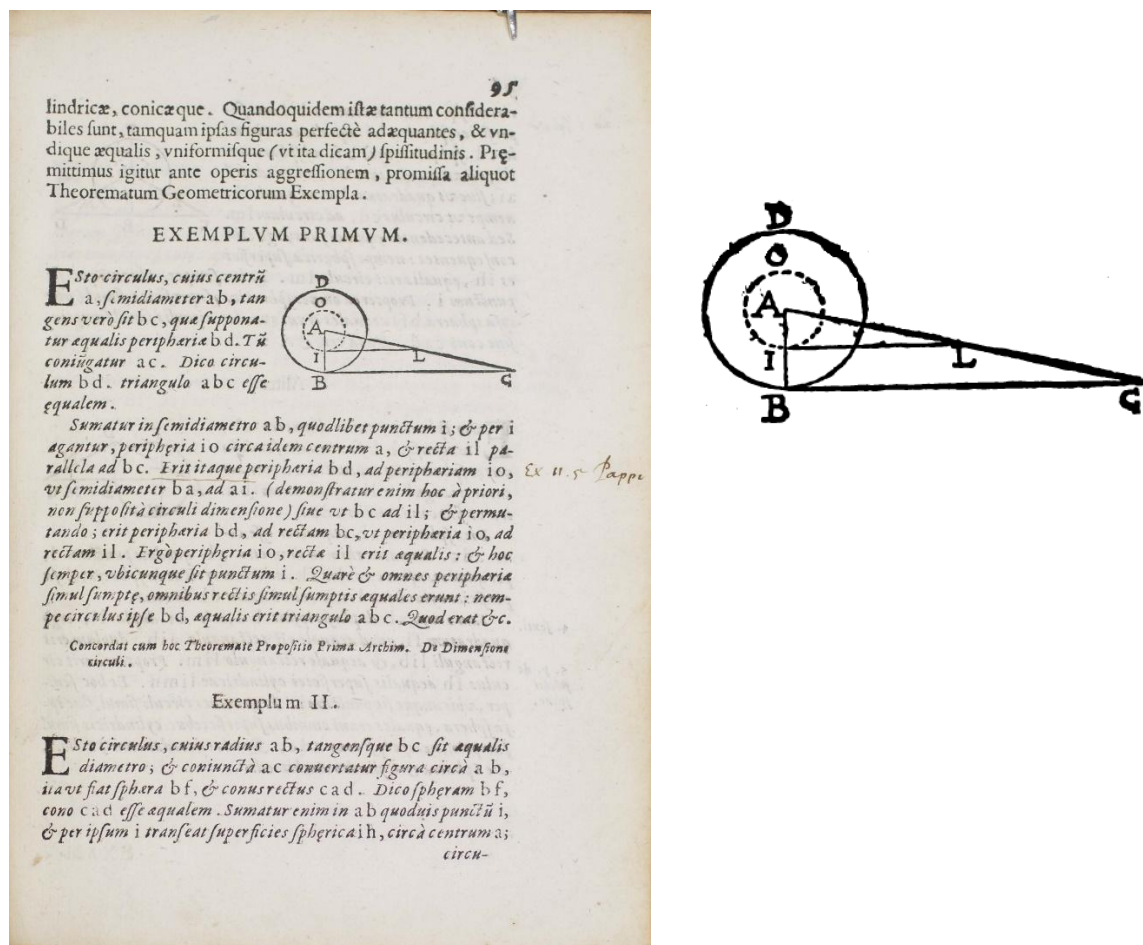


Fig. 40. The diagram used by Torricelli to square the circle through the curved indivisibles.
(Torricelli 1644, IV, p. 95; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 970-971)
Source: Courtesy University of Oklahoma, History of Science Collections.

Torricelli reasons like this: on the radius AB , take a point I . Draw the circumference of radius AI and the segment IL parallel to BC , which is the tangent at B . Choose BC as long as the circumference BD . The circumference BD is to the circumference IO as AB is to AI ⁵⁴, that is as BC to IL . Therefore $\text{circ. } BD : BC = \text{circ. } IO : IL$. Thus, $\text{circ. } IO = IL$. Hence, all the circumferences constructed as IO , which compose the circle BD are respectively equal to all the segments as IL which compose the triangle ABC . Ergo, the area of the circle and that of the triangle are equal. The circumference IO is one of the curved indivisibles which compose the circle of diameter BD .

The XIIth example concerns the volume of the paraboloid of revolution (Fig. 41).

⁵⁴ Torricelli points out that this property does not depend on the quadrature of the circle (ibid., p. 95).

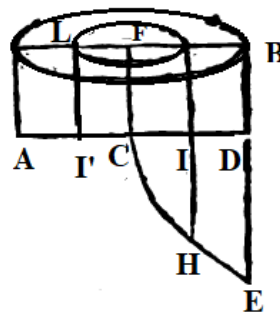
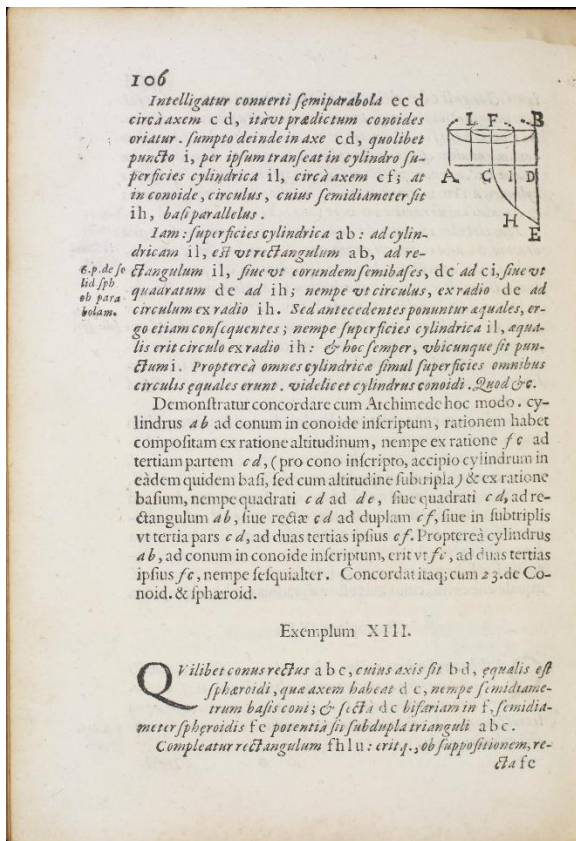


Fig. 41. Adapted. The diagram used by Torricelli to prove the example referred to in the running text. For convenience of the reader, we have completed the figure (right side of the table).
(Torricelli 1644, IV, p. 106; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 992-993)
Source: Courtesy University of Oklahoma, History of Science Collections.

Consider the half-parabola EH of height DC . Take $AC=DC$. Construct DB so that $ED = \sqrt{AD \cdot BD}$. The theorem states that the right cylinder ADB of axis CF has the same volume as the paraboloid obtained by the rotation of the parabolic segment CED around the axis CD . Take a point I on CD , so that the cylindrical surface IL passes through I as well as the semi-circle of radius IH . The cylindrical surface AB is to the cylindrical surface IL as $AD \times DB$ is to $I'I \times DB$, namely as DC is to IC and as ED^2 is to IH^2 . Therefore, since

$$ED^2 = AD \times BD, \text{ it is } IH^2 = I'I \times DB.$$

This means that the cylindric surfaces, which are the curved indivisibles that compose the cylinder, are equal to the circles which compose the paraboloid. Thus, the theorem is proved. Afterwards, Torricelli proves that this theorem agrees with Proposition XXIII of Archimedes's *Conoids and Spheroids*.

After the examples, Torricelli defines the "acute hyperbolic solid" as the figure generated by the rotation of a hyperbola around one of its asymptotes (Torricelli 1644, IV, p. 112). The fundamental theorem where Torricelli determines the volume of the acute hyperbolic solid is preceded by five lemmas. Their proofs are relatively clear and do not need any particular demonstrative technique as indivisibles or exhaustion.

Lemma I (Fig. 42) considers a hyperbola, whose asymptotes AB and AC are perpendicular. Rotate the figure around the asymptote AB , so generating the acute hyperbolic solid. Inscribe any rectangle $DEFG$ passing through the asymptote in the solid. This rectangle is equal to the square of the hyperbola's semi-axis.

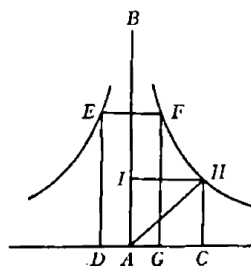


Fig. 42. The diagram used by Torricelli to prove the Lemma I, but usable also in reference to the other Lemmas. (Torricelli 1644, IV, p. 113; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 1006-1007)

Source: Courtesy University of Oklahoma, History of Science Collections.

In Lemma II (Fig. 42) Torricelli proves that all the cylinders described in the acute hyperbolic solid around the rotation axis AB have the same external surface.

In Lemma III it is proved that the volumes of such cylinders are as the diameters of the bases.

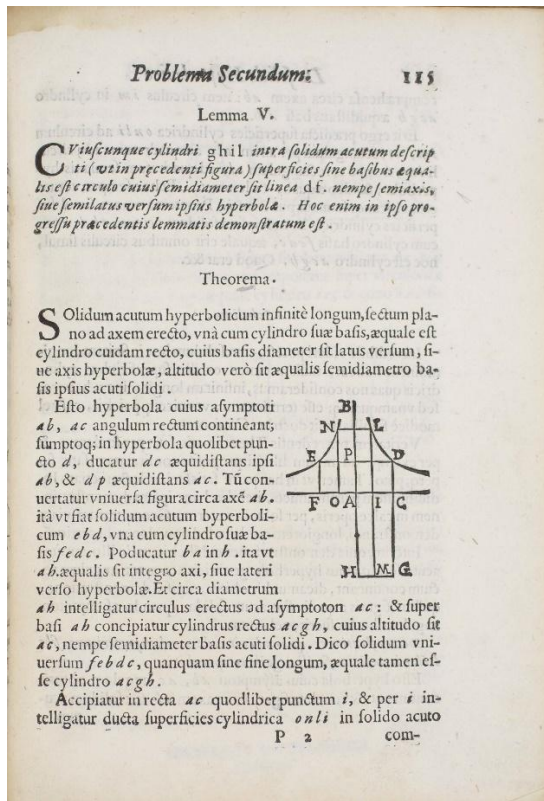
In Lemma IV Torricelli demonstrates that the lateral surface of the cylinder $DEFG$ is $\frac{1}{4}$ the surface of the sphere whose radius is equal to the hyperbola's semiaxis.

Finally, Lemma V (Torricelli 1644, IV, p. 115; see also Fig. 37), which derives directly from the previous one, claims that the lateral surface of the cylinder $DEFG$ is equal to that of the circle whose radius is the hyperbola's semi-axis AH . Armed with the concept of curved indivisibles and with these five lemmas, Torricelli is able to determine the volume of the acute hyperbolic solid:

Consider the infinitely long acute hyperbolic solid cut with a plane perpendicular to the axis. Its volume plus the volume of its base's cylinder is equal to the volume of the right cylinder whose basis is the versed side, namely the hyperbola's axis, and whose height is equal to the radius of the acute solid's base.⁵⁵

⁵⁵ "Solidum acutum hyperbolicum infinite longum, sectum plano ad axem erecto, una cum cylindro suae basis, aequale est cylindro cuidam recto, cuius basis diameter sit latus versum, sive axis hyperbolae, altitudo vero sit aequalis semidiametro basis acuti solidi." (Torricelli 1644, IV, p. 115; Authors' translation).

Thus, Torricelli proved that (Fig. 43):



Be given the hyperbola whose asymptotes AB and AC are perpendicular. Consider any point D on the hyperbola. Draw ED parallel to AC and DP parallel to AB . Rotate the figure around the asymptote AB as rotation axis. The acute hyperbolic solid EDB is generated. It has the circle of diameter ED as base and is infinitely extended along PB as axis. Now produce AB until H so that AH is equal to the hyperbola's axis. On the base AH construct the right cylinder $AHGC$, whose height is AC . The volume of the acute hyperbolic solid plus that of the cylinder $EDCF$ is equal to the volume of the cylinder $AHGC$.

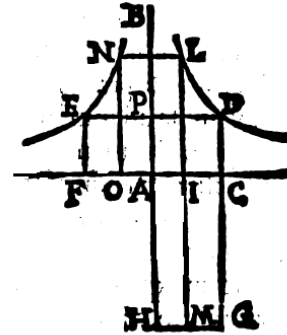


Fig. 43. The determination of the volume of the acute hyperbolic solid.

(Torricelli 1644, IV, p. 115; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 1008-1009)

Source: Courtesy University of Oklahoma, History of Science Collections.

In order to prove the theorem Torricelli reasons like this: on AC he considers any point I ; he draws the parallel to AB which cuts the hyperbola in L and constructs the rectangle of dimensions AI and IL . If this rectangle rotates around AB , the cylindrical surface $ONLI$ is generated. In the cylinder $ACGH$ consider the circle of diameter IM , where M is obtained producing IL . The cylindrical surface $ONLI$ is to the surface of the circle whose diameter is to as in the following:

$$IM \text{ as } OI \times IL \text{ is to } \left(\frac{IM}{2}\right)^2.$$

Because of Lemma V these surfaces are equal, for any position of the point I . Thus, considering the acute hyperbolic solid plus the cylinder at its base composed of indivisibles such as the cylinder $ONLI$, while the cylinder $ACGH$ is composed of the circles whose diameter is IM , it follows that the volume of the two figures is equal. This fundamental theorem is followed by 29 corollaries where Torricelli proved a series of properties connected to the acute hyperbolic solid. The corollaries are preceded by a brief but interesting scholium where Torricelli expresses his astonishment for the result he had obtained. For, we read:

It might appear unbelievable that, though this solid having an infinite length, no one of the considered cylindrical surfaces has an infinite length. All of them are finite, as each one having a mediocre familiarity with the doctrine of conic sections will clearly recognize.⁵⁶

⁵⁶ "Incredibile videri potest, cum solidum hoc infinitam longitudinem habeat, nullam tamen ex illis superficiebus cylindricis quas nos consideramus, infinitam longitudinem habere; sed unamquamquamque esse

These words are similar to those written by Cantor to Dedekind in a letter on 29 June 1877 after having discovered the bijection between the bi-dimensional and the linear *continuum*: “I see it, but I do not believe it”⁵⁷.

Cavalieri, as the Dedekind of Torricelli, shared the astonishment for this discovery and expressed his admiration for Torricelli in a letter written on 17 December 1641 where we read:

I received your letter when I was in bed with fever and gout. [...] Despite my sickness, I was able to enjoy the very tasty fruits of your ingenuity. I came to be in infinite admiration of that infinitely long hyperbolic solid, which is equal to a three dimensional finite body. When I communicated this result to some of my disciples, who are philosophers, they admitted this fact to appear them really marvellous and astonishing. [...]. I cannot understand how you were able to guess [literally to fish] so easily, in the infinite depth of that solid, its dimension. Indeed, the solid seems to me infinitely long, because the plane space which generates the solid appears to me infinitely long and each part of this plane space generates a part of the solid.⁵⁸

Torricelli does not miss the occasion to confirm that the demonstration through indivisibles is perfectly legitimate from the geometrical standpoint. However, he also added a proof by exhaustion to satisfy the reader who raises doubts on the indivisibles (Torricelli 1644, III, 2, p. 116).

In the second and last part of his work, entitled, *De Dimensione Acuti solidi Hyperbolici iuxta methodum Antiquorum* (Torricelli 1644, IV. The numbering of pages restarts from 129. Thus it should be: p. 136 and pp. 129-135) Torricelli determines the volume of the hyperbolic solid through exhaustion. In the brief introductory consideration, he claims that the most difficult aspect of solving such a problem by exhaustion might appear the inscription and circumscription of a figure to a figure of infinite length. However, Torricelli, recalls that Roberval, too, solved such a problem with methods different from his own. He had received Roberval's letter on the cubature of the hyperbolic solid when the *Opera geometrica* was already in press, so that he could not publish Roberval's proof. Furthermore, from Roberval's letter, it was not clear whether he desired his proof to be published.

terminatam; ut unicuique patebit, cui vel modice familiaris sit doctrina Conicorum.” (Torricelli 1644, IV, p. 116; Authors' translation).

⁵⁷ In his letter to Dedekind, which obviously is written in German, Cantor wrote in French (to underline even more his astonishment for the result he had obtained): “[...] je la vois, mais je ne la crois pas [...].” (Cantor and Dedekind 1937, p. 34). After few days, on 2 June 1877, Dedekind answered to have found no lack or mistake in Cantor's proof. He wrote: “I developed once again the proof of your demonstration and I found no lack. I think your interesting proposition to be correct and I congratulate you” (Authors' translation; German text: “Ihren Beweis habe ich noch einmal geprüft, und ich habe keine Lücke darin entdeckt; ich glaube gewiss, dass Ihr interessanter Satz richtig ist, und beglückwünsche Sie zu demselben.” (Ivi, p. 37).

⁵⁸ “Mi giunge la lettera di V.S.M. Rev.do in tempo che io stavo nel letto con la febbre, e gotta [...] ho però goduto al dispetto del male d' saporitissimi frutti del suo ingegno, essendomi riuscito infinitamente ammirabile quel solido iperbolico infinitamente lungo, ed uguale ad un copro, quanto a tutte e tre le dimensioni, finito, ed avendolo io comunicato ad alcuni miei scolari filosofi, hanno confessato parergli veramente meraviglioso, e stravagante, che ciò possa essere [...] non so come abbi pescato nell'infinita profondità di quel solido così facilmente la sua dimensione, poiché veramente a me pare infinitamente lungo, parendomi infinitamente lungo lo spazio piano, che lo genera, ad ogni parte di esso generando parte del solido”. We point out that the term “filosofo” was then used also to indicate the mathematicians and the natural philosophers, namely the physicists. (Torricelli 1919-1944, III, pp. 65-66; Authors' translation. See also Belloni's Introduction in Torricelli 1975, p. 14; Gàbici and Toscano 2006, pp. 56-57).

Torricelli's proof is based upon seven lemmas. The most interesting of these lemmas is the sixth one, the only one proved by exhaustion (Fig. 44):

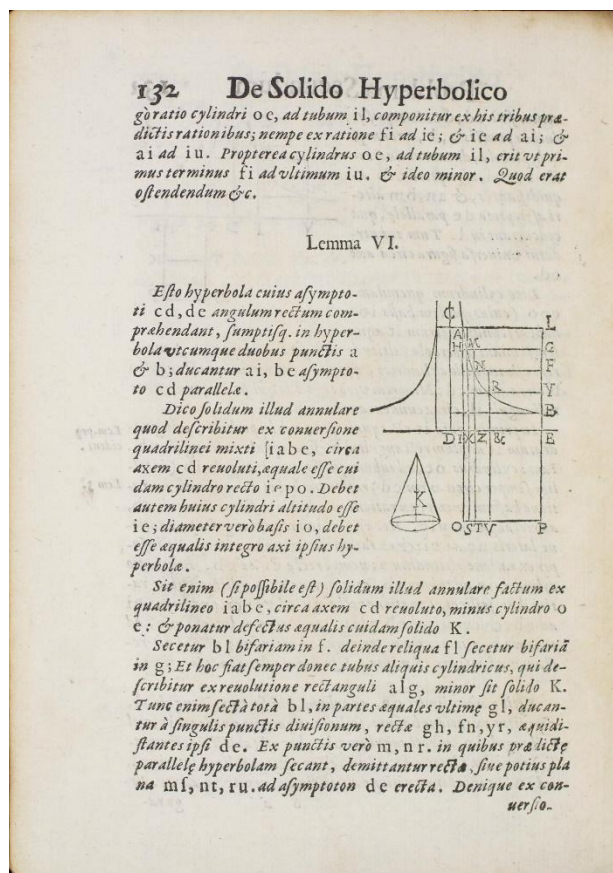
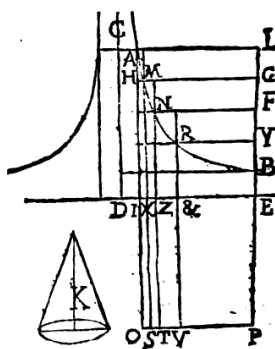


Fig. 44. Lemma VI and its diagram

(Torricelli 1644, IV, p. 132; Cf. Pisano, Dhombres, Radelet de Grave and Bussotti, 2024, pp. 1060-1061)

Source: Courtesy University of Oklahoma, History of Science Collections.

Be given a hyperbola whose asymptotes CD and EF are perpendicular. Take any two points A and B on the hyperbola and draw the parallels to CD . Rotate the mixtilinear quadrilateral $IABE$ around CD . The so obtained hollow solid has the same volume as the right cylinder $IEPO$.



The proof of the theorem runs like this (Fig. 44): suppose the hyperbolic solid to be less than the cylinder $ACIH$, constructed as the cylinder $ACGH$ of previous Fig. 38. Consider the cylinder $NCIL$ equal to the hyperbolic solid. Produce LN until its intersection M with the hyperbola. The point M exists since LN is supposed parallel to the asymptote AB . Because of Lemma VI, the cylinder $NCIL$ is equal to the hollow solid described by the rotation of the mixtilinear quadrilateral $NMDC$ around AB . Therefore, it will be less than the hyperbolic solid, which is against the assumption. In a similar manner, Torricelli proves that the cylinder $NCIL$ cannot be less than the hyperbolic solid. The demonstrative core of this theorem is Lemma VI proved by exhaustion.

Torricelli's *Opera geometrica* ends with an appendix concerning several geometrical properties of the cochlea⁵⁹.

The main purpose of our Introduction is to explain Torricelli's results obtained in *Opera geometrica*. We have expounded them and commented on the most important methods he used. We have also seen what the similarities and the differences between exhaustion method and integral calculus are. Given their relevance for the history of geometry, we wish to offer the reader a view on two other important mathematical discoveries by Torricelli and add some considerations on indivisibles in relation to integral calculus. Thus, we will briefly analyse two works which do not

⁵⁹ As above cited, *Opera Geometrica* often restarts the order of the numbering pages. When it comes to *Appendix De dimensione Cochlea*: pp. [136]145–150. The numbering of pages starts with p. 136 and just after with p. 145, so the numeration continues consequently. The last page is 150, but, in fact, the appendix is only seven pages long.

belong to the *Opera Geometrica*: the memoir on the logarithmic spiral and that on the infinite hyperbolas.

4. Infinite Curves

This section is devoted to analysing and commenting upon relationship between Torricelli's geometrical methods and infinitesimal calculus. We will do it by means of two – at that time – unpublished works of Torricelli (both in Torricelli 1919–1944; *De Infinitis Spiralibus* in Torricelli 1975, pp. 507–548). In fact, these manuscripts were not included in *Opera geometrica* and neither were they published during Torricelli's lifetime. This adequately allows us to introduce Torricelli's calculus with infinite and infinitesimal magnitudes in the next section below.

A geometrical (also logarithmic) spiral is represented by a curve, which originates from a point, moving farther away as it revolves around the point (logarithmic spiral and geometrical properties). The argument is geometrically complex and fascinating. The traditions of works on spirals and their attractive representation in nature run from Antiquity to Fibonacci *et al.* (Pisano and Bussotti 2015c).

First, below we present *De infinitis spiralibus* (1645). It represents advanced research on spiral curves dealt by Torricelli. Two of the important goals achieved by Torricelli were: 1) the elegant dimensional passage to geometrical approach, to logarithmic spiral and 2) to have made clear its difference from two-dimensions Archimedean spiral (although Torricelli was inspired by it which consisted of equal arms only). We also remark the intellectual charm of these asymmetric curves, as presented by Torricelli in their logarithmic forms. It is important to note the infinite character of the logarithmic spiral, which, along with its generation around a fixed center–point, approaches or moves away from this center–point without ever reaching it. Thus this center–point can be considered as an asymptotic point. In modern terms, we could say that Torricelli pointed out that even if the geometry of the problem is typically central–asymptotic, that is it is necessary to rotate infinite times around the center before reaching it, the total distance travelled is instead finite. Therefore, the intellectual charm consisted in having shown that for the logarithmic spiral the *infinite* and the *finite* coexisted at least in mathematical reasoning. It is evident that *De infinitis spiralibus* belonged with Torricelli's *Opera geometrica* arguments as part of his concurrent research program, even though, it was not published in 1644 with *Opera geometrica*.

Secondly, we will deal with another outstanding work: *De infinitis hyperbolis* (1646). We will analyse this interesting geometrical—mathematical essay as a transcript into modern language by Bortolotti from Torricelli's original work. Taking into account Bortolotti's arguments, below we propose our own standpoint on the subject. We will show how Torricelli developed a calculus with infinite and infinitesimal quantities. But, in our opinion, this Torricellean calculus cannot be considered as the precursor of infinitesimal calculus. For, we will point out several aspects which make Torricelli's work irreducible to the mentality and to the concepts which, a few years later, connoted infinitesimal calculus.

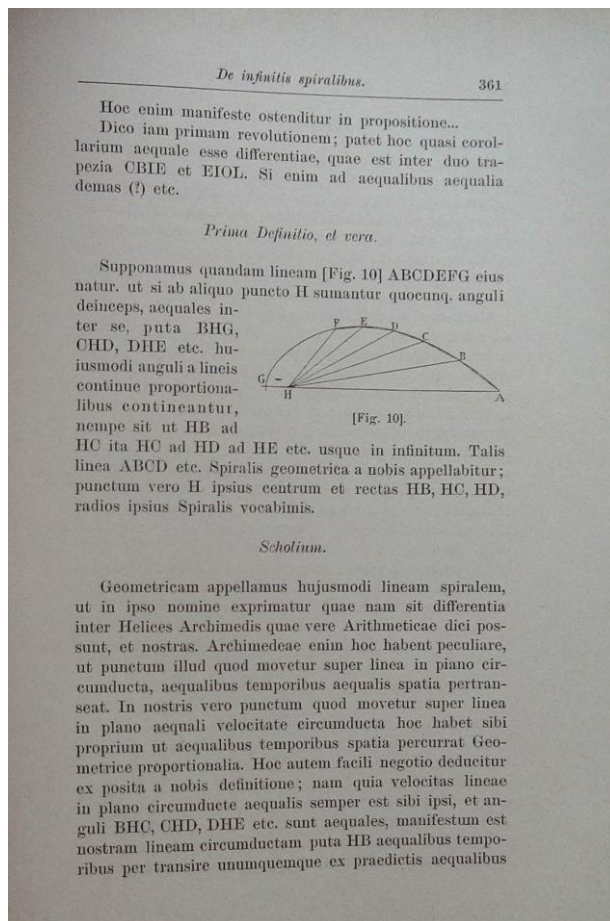
4.1 *De infinitis spiralibus* (1645)

Torricelli's *De infinitis spiralibus* (1645) is an important geometrical essay where Torricelli solved very difficult problems. It was posthumously published both in *Opere di Evangelista Torricelli* by Gino Loria and Giuseppe Vassura (Torricelli 1919–1944, I, 2, pp. 349–399) and in *Opere scelte di Evangelista Torricelli* (Torricelli 1975, pp. 507–548) by Lanfranco Belloni. The latter is based on Carruccio's previous work (Torricelli 1955, 1975, p. 49, p. 488; see also Bortolotti 1925ab, 1928b, 1939). As far as the two editions are almost equivalents, at first sight we follow the former (Torricelli 1919–1944). However, when they diverge, we follow the reconstruction by Carruccio (1955) then edited by Belloni (Torricelli 1975, pp. 507–548).⁶⁰

⁶⁰ It is to be noted out that Descartes already introduced the logarithmic spiral. In a letter to Mersenne written on 13 July 1638 Descartes deals with a problem of physics (Descartes 1897–1913, II, pp. 222–245). In this context Descartes introduces the logarithmic spiral (*Ivi*, pp. 232–233). But in this letter there is only a hint of the curve. By far more important is the letter Descartes sent to Mersenne on 12 September 1638 (*Ivi*, pp. 352–362). Descartes claimed that “[...] this spiral has many properties which allow us to recognize it [...]” (“Mais, pour cette spirale, elle a plusieurs propriétés qui la rendent assez reconnaissable”; *Ivi*, p. 360). In particular: 1) if *A* is

In the following we refer to Loria and Vassura's edition of Torricelli's *Opere* (Torricelli 1919-1944).

In the *De infinitis spiralibus* Torricelli describes (Fig. 40) the following curve, introducing its “First and true definition” (*Prima Definitio, et vera*; Torricelli 1919-1944, I, 2, p. 361):



Consider the line such that, if from a point H equal angles $BHC, CHD, DHE, \dots$ are drawn, such angles have the sides in continued proportion. Namely it holds: $HB:HC = HC:HD = HD:HE, \dots$ Such line is named by Torricelli a geometrical spiral (*spiralis geometrica*). The point H is its centre and the segments $HB, HC, HD, \dots$ its radii.

Fig. 45. Torricelli's diagram on *Prima Definitio, et vera* (Torricelli 1919-1944, I, 2, p. 361; see also Torricelli 1975, p. 511).
Source: Public domain

The name chosen by Torricelli depends on a kinematic property of his curve, as he clarified in the scholium posed after the definition: the Archimedean spiral can be regarded as generated by a point moving uniformly on a straight line rotating uniformly around a centre. This point, hence, describes equal spaces in equal times, whereas in the geometrical spiral the spaces traversed in equal times are in geometrical proportion. This means that the point moves on the straight line of uniformly accelerated motion. This argument was clearly proved by Torricelli as follows: the straight line GA rotates uniformly and traverses equal angles in equal times. The sides of the angles are in continuous proportion, so that their differences are, too. But the differences of the radius vectors represent the spaces. Therefore, Torricelli gives a geometrical definition of his spiral from which its kinematic characterization also follows. The profound research by Torricelli continued showing a further fundamental difference between his spiral and Archimedes's one: if, in the latter, you consider a point P , the number of turns is finite towards the origin of the curve and infinite in the opposite direction.

the origin of the spiral and B_i its points, there is the same proportion between the arch of spiral AB_i and the radius vector AB_i , which is a remarkable discovery; 2) the angle between the tangent and the radius vector is constant for every point of the spiral (*Ibidem*). There is, however, no demonstration or systematic study of this curve in Descartes, although he recognised some of its main features. It is well known that, after Torricelli, many mathematicians studied the equiangular spiral, first of all Jakob Bernoulli, who defined it *spira mirabilis*. However, in our context we do not think appropriate to face this question.

Whereas in the geometrical spiral, the number of turns is infinite in both directions. Torricelli gives this brilliant proof (Fig. 46):

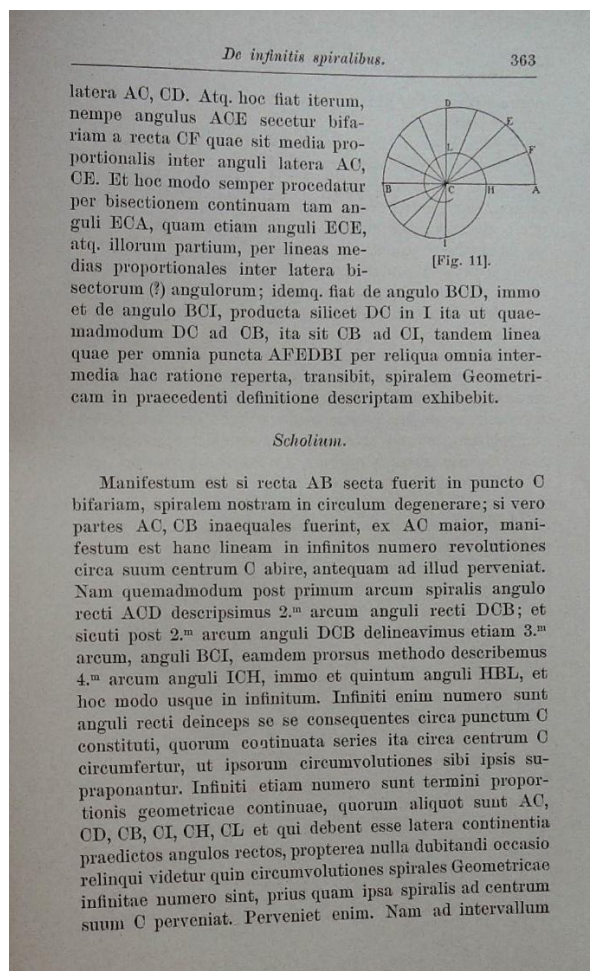


Fig. 46. Torricelli's Proof which show the difference between his spiral and Archimedes's one (Torricelli 1919-1944, I, 2, p. 363; see also Torricelli 1975, p. 514).

Source: Public domain

Now there is a noteworthy reasoning: Torricelli claims that the spiral, after an infinite number of rotations will reach the centre *C*. He argues like this: The reasoning previously shown proves that the spiral will arrive closer to the centre than any given finite distance. Thus, it will reach the centre. He argues:

It reaches the centre: for, it arrives closer to the centre than any given interval. Therefore, it reaches the centre itself.⁶¹

Torricelli approached the concept of limit because, to use modern terms, he considers the limit of the spiral when the number of turns tends to infinity, and claims, correctly that such a limit is the centre of the spiral. On the other hand, he also concludes that the limit-value is reached at the end of the process, which is not necessarily true. Thus, we think one can legitimately conclude that Torricelli grasped many aspects of the concept of limit, but some important features escaped him.

Torricelli proves several fundamental geometrical properties of the logarithmic spiral: first of all, he demonstrates that, considering two points of the spiral, the straight line joining them is internal to the spiral. This implies that the curve has a tangent in every point. Afterwards, Torricelli proves two

⁶¹ "Perveniet enim. Nam ad intervallum perveniet quolibet dato intervallo minus; ergo et ad ipsum centrum perveniet." (Torricelli 1919-1944, I, 2, pp. 363-364; Authors' translation).

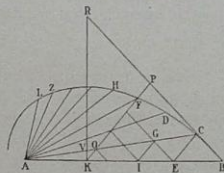
NO, OL, sive quam DE, OL; sumpta communi LE, erit tota tangens CI maior quam DE, EL, LO. Sed DE, EL, LO maiores sunt, per suppositionem quam arcus OHD; ergo ipsa tangens IC multo maior erit quam arcus OHD. Quod etc.

Recta BR aequalis perimetro rectilineo BCL.

Esto spiralis cuius centrum [Fig. 18] A, maximus vero radius AB, et sumantur ad centrum A quotlibet anguli deinceps aequales BAC, CAD, DAF etc., compleanturq. triangula BAC, CAD, DAF etc. Secentur AV et AK aequales AL minimo siliet lateri triangulorum, et iungatur KV quae producta concurret cum BC producta in R. Dico rectam BR aequalem esse omnibus inscriptis BC, CD, DE, etc. usq. ad punctum L simul sumptis.

Cum enim propter aequales angulos ad centrum sumptos sit ut BA ad AC, ita AC ad AD, erunt triangula BAC, CAD, et reliqua etiam omnia usq. ad ultimum inter se similia. Si vero secentur aequales AE ipsi AD, AH ipsi AF, AI ipsi AH, et hoc modo semper usq. ad ultimam AK quae iam facta fuit aequalis ipsi AL, erunt, per quartam Primi Euclidis, aequales inter se CE, CD; aequalesque inter se EG, DF, et inter se GI, FH, et reliquae omnes usque ad ultimam AL quae aequalis erit ultimae OK. Erunt etiam parallelae omnes si alternatim sumantur, nempe BC, EG et reliquae alternatim sumptae, itam CE, GI et reliquae alternatim sumptae inter se parallelae erunt. Nam BA ad AC est ut AC ad AE, sive ut AE ad AG, quamobrem parallelae sunt BC, EG, et sic de reliquis. Producat iam ultima KO in P, et erit ipsa KP aequalis universo ordini parallelarum CE, GI etc. Patebit hoc si producantur EG cum omnibus reliquis sibi parallelis usque ad KP, dividant

Ut p. a. diff. ad omnes diff. ita pri- mus terminus ad omnes simul terminos. Si vide in questa figura, excepto ultimo, sive minimo; non i radij sono li e lo applicate sono 8. Facile estenditur ex 12 Quinti.



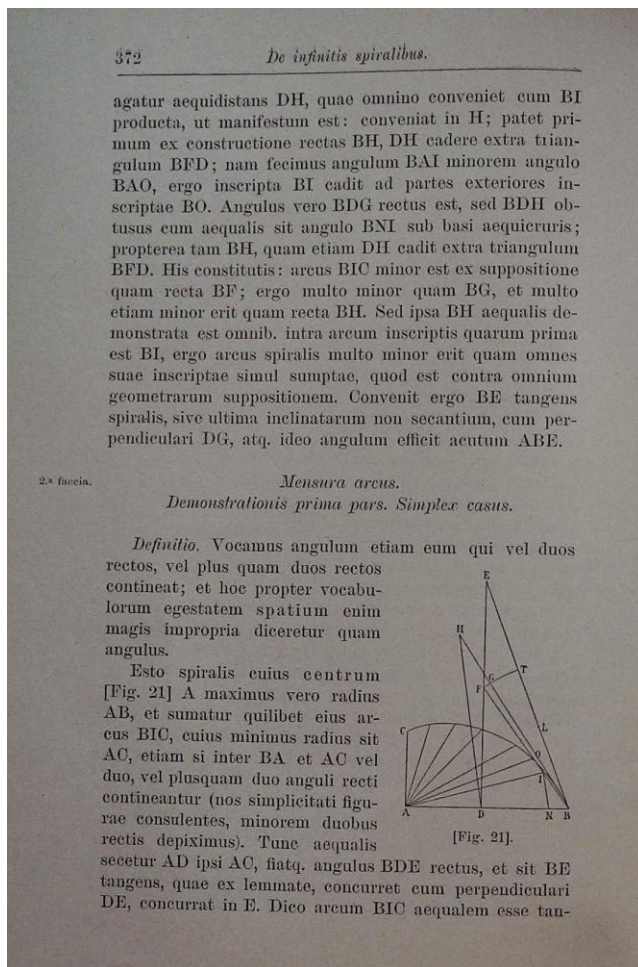
[Fig. 18].

Be AB the longest radius of the spiral sector BL . From the centre A draw any number of equal consecutive angles $BAC, CAD, DAF, \dots$, so as to obtain the triangles $BAC, CAD, DAF, \dots$. On AB and AC take the segments AK and AV equal to the smallest radius AL . The segment VK produced cuts the segment BC produced in R . The segment BR is equal to the sum of the inscribed polygonal $BC + CD + \dots + ZL$.

Fig. 48. Torricelli's diagram on a fundamental property of the logarithmic spiral.
(Torricelli 1919-1944, I, 2, p. 369; see also Torricelli 1975, p. 523)

Source: Public domain

The rectification of the geometrical spiral runs like this (Fig. 49):



Be A the centre of the spiral; AB the longest and AC the shortest radius of the considered arch. Be $AD=AC$. Construct the right angle BDE . Be BE the tangent, which, for a previous Lemma, cuts DE . The theorem claims that the arch BC is equal to BE . (With BC we will indicate the arch of spiral).

Fig. 49. Torricelli's diagram on the rectification of the geometrical spiral.
(Torricelli 1919-1944, I, 2, p. 372; see also Torricelli 1975, p. 529).
Source: Public domain

Torricelli speaks of two proofs, one by exhaustion and the other by indivisibles. However, it was possible to find in Torricelli's manuscripts only the proof by exhaustion.

It is so systematically conceived:

suppose $BC < BE$.

From BE subtract $BL=BD$.

Bisect continuously LE until you obtain a segment $TE < BE - BC$.

Therefore $BT > BC$. From B draw BF , such that F belongs to DE and $BF = BT$.

Therefore, by construction $BC < BF < BE$.

Since BE is the tangent, BF cuts the spiral in O . Join A and O .

Bisect continuously the angle BAC until reaching the angle $BAI < BAO$.

Consider the polygonal inscribed in the spiral whose first side is BI , which, produced, belongs to the angle EBF . On AB take $AN = AI$. Join N and I .

Draw DH parallel to IN . The straight line DH is external to the angle EDB . For, EDB is right, while HDB is obtuse because it is equal to INB ($DH \parallel IN$), which is adjacent to the base of the isosceles triangle ANI .

Thus, DH and BI cut in H . Since $BF > BC$, it is also $BG > BC$ and $BH > BC$.

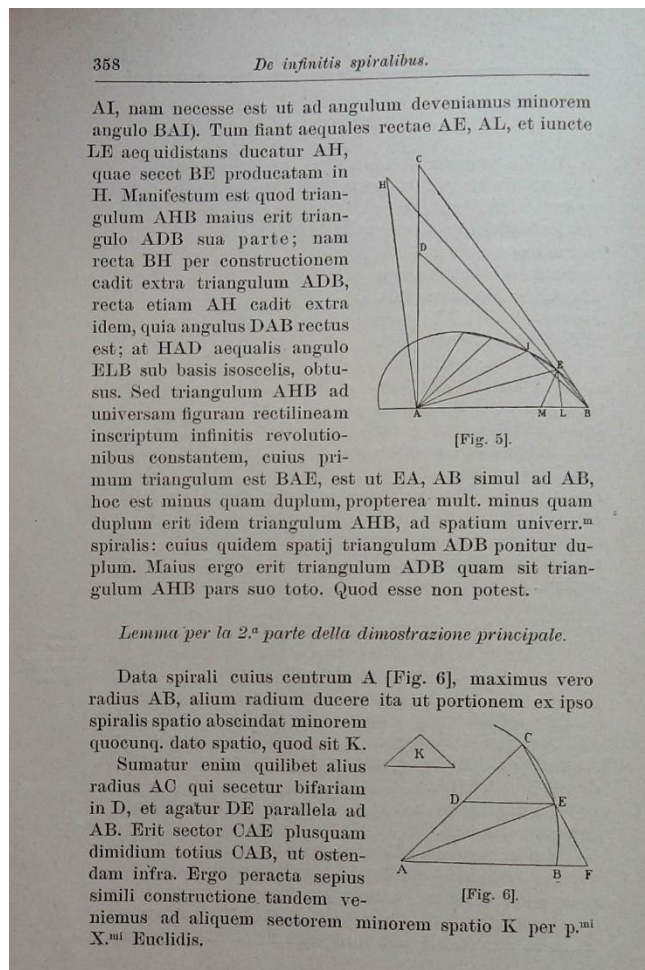
However, the segment BH , because of the previous theorem, is equal to the sum of the inscribed polygonal, whose first side is BI .

Hence, the sum of the sides of the inscribed polygonal should be bigger than the arch of spiral, but this is absurd.

A consequence of Torricelli's theorem is that any segment of the geometrical spiral has a finite length, even when it includes the infinite turns of the spiral around its centre. With a similar reasoning Torricelli proved that neither is the arch bigger of the considered tangent segment.

In other parts of the *Infinite spirals*, Torricelli proves, through exhaustion, another astonishing result concerning the logarithmic spiral: given a spiral segment, whose perimeter has the centre as an extreme and a point on the spiral as the other extreme, the area of the segment is finite, despite the infinite turns of the spiral. The road that led Torricelli to the proof of his theorem is rather complex: first he proves some theorems on the areas of polygonal figures inscribed in the spiral and on the areas of particular triangles, one side of which is a segment of a tangent to the spiral.

Below we present Torricelli's interesting result on the quadrature of a surface. He proved a theorem with regard to the area of a spiral (Fig. 50).⁶²



Be A the centre of the spiral, AB the longest radius of the segment, and BC the tangent to the spiral. Draw AC perpendicular to AB . The whole surface including infinite revolutions of the spiral is equal to the triangle ABC .

Fig. 50. Torricelli's diagram (bottom page) on the theorem with regard to the area of the spiral. (Torricelli 1919-1944, I, 2, p. 358; see also Torricelli 1975, I, p. 542).

Source: Public domain

⁶² We refer to Belloni, who based his editing of *De infinitis spiralibus* (Cfr. Torricelli 1975, p. 488) on Carruccio's work. Both were notable Torricellean scholars and editors (see also: Bortolotti 1925ab, 1928b, 1939). Historiographically interesting, for the editorial projects of Torricelli's *Opere* during the past century, is Belloni's remark to Loria about *De infinitis spiralibus* (Torricelli 1975, p. 488; see also Belloni's tale on Bortolotti's attack on Loria, *Ivi*, p. 50 [pp. 49-51]). We continue to use Loria and Vassura's images (Torricelli 1919-1944). We note that Loria and Vassura presented the image and its content (Fig. 45) before (Torricelli 1919-1944, I, 2, p. 358) the theorems discussed above (Figs. 40-44; *Ivi*, p. 361, p. 363, p. 366, p. 369, p. 372). Instead, Belloni-Carruccio provided it after (Torricelli 1975, p. 542) the above-cited theorems. We also point out a drawing-difference between Loria and Vassura's figure (Fig. 45) and Belloni's one (Torricelli 1975, p. 542). The reader is invited to pay attention to the point H in the two figures.

We do not reproduce the details of Torricelli's proof here, because we believe the details furnished so far are sufficient for the reader to gain an idea of Torricelli's methods and way of thinking. His two key methods are exhaustion and indivisibles, which he uses with great ability and extends to an extraordinary number of cases which prior to his work were not included in the range of these two procedures. The second aspect to underline is his skill in treating questions concerning infinity. He had no fear of facing figures which, in different manners, entail infinity and to find geometrical means to treat them. One might say that Torricelli was one of the first mathematicians to measure infinity and to discover many unexpected properties such as those relative to the volume of the acute hyperbolic solid or to the rectification and quadrature of the geometrical spiral. Finally, we offer some results obtained by Torricelli in his memoir *De infinitis hyperbolis* (Torricelli 1919-1944, I, 2, pp. 227-274) jointly with the commentaries by Bortolotti, because this will allow us to develop a brief discussion on Torricelli's methods.

4.2. *De infinitis hyperbolis* (ca. 1646)

The idea behind this *De infinitis hyperbolis*, is to study the curves which, in modern terms, are written as $x^m y^n = k$, where m and n are integers. Without limitation, it is possible to suppose $|n| > |m|$. Torricelli, however, always refers to specific cases and never resorts to algebraic methods, but to geometrical ones. The memoir was far from being ready for publication. Therefore, the material is not well ordered and it is necessary to offer a line of interpretation. It is useless to present the propositions one after the other. In the following we follow Belloni (and Carruccio) as Torricelli's *Opere* editors and Bortolotti (Bortolotti 1925ab, 1928b, 1939; see also Torricelli 1919-1944, pp. 231-274).

Let us start to follow the excellent line of interpretation offered by Bortolotti. It is possible to identify a fundamental lemma, which, Bortolotti transcribes like this in modern terms (Bortolotti 1925b, pp. 946-948; see Fig. 51):

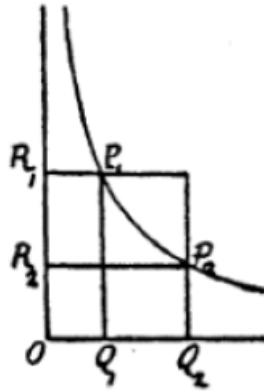


Fig. 51. The diagram used by Bortolotti to prove Torricelli's fundamental Lemma (Bortolotti 1925b, p. 946).
Source: Public domain

Be given the curve $x^m y^n = k$. Consider two points $P_1 \equiv (x_1, y_1), P_2 \equiv (x_2, y_2)$ belonging to it. Be $Q_1, Q_2; R_1, R_2$ their projections on the axes. Complete the rectangles $P_1 Q_2, P_2 Q_1, P_1 R_2, P_2 R_1$.

The Lemma can be divided in three main parts as in the following:

- 1) The ratio of the inscribed triangles is greater than that of the exponents, which is greater than those of the circumscribed triangles. Namely:

$$\frac{P_2 Q_1}{P_1 R_2} > \frac{n}{m} > \frac{P_1 Q_2}{P_2 R_1}$$

- 2) Rotate the figure around the y -axis. The ratio of the solids generated by the rotation of the inscribed rectangles, the exponents' ratio and that of the solids generated by the rotation of the circumscribed rectangles satisfy the following law:

$$\frac{\text{solid generated by } P_2 Q_1}{\text{solid generated by } P_1 R_2} > \frac{2n}{m} > \frac{\text{solid generated by } P_1 Q_2}{\text{solid generated by } P_2 R_1}$$

- 3) Analogous relations hold if the rectangles rotate around the x -axis.

Bortolotti (Bortolotti 1925b) transcribes the geometrical reasonings into algebraic and general terms reaching the demonstrations in few passages. However, Torricelli analyses some specific cases and offers geometrical proofs which are rather complex. The most complete of these proofs is relative to the hyperbola

$$xy^3 = k.$$

It is useful to report it for the reader to understand the profound difference between Torricelli's way of reasoning in respect to an algebraic approach (Fig. 52). In the following we come back to refer to Loria and Vassura as Torricelli's *Opere* editors (Torricelli 1919-1944).

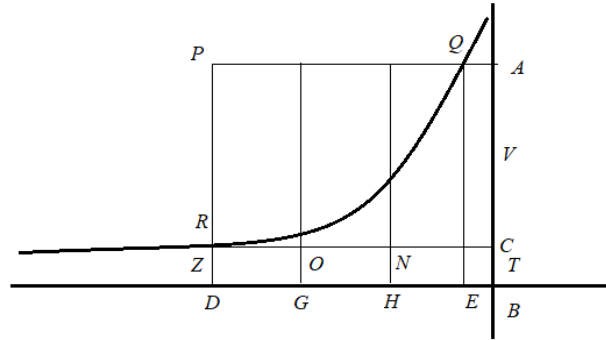


Fig. 52. Our representation of the figure used by Torricelli to explain the reasoning above argued. (Torricelli 1919-1944, I, 2, p. 246).

Consider two points $Q \equiv (E, A); R \equiv (D, C)$ belonging to the hyperbola. It is $AB^3:BC^3 = DB:BE$. Consider two mean proportionals GB, HB between DB and EB . Since it holds

$$DB:GB = GB:HB = HB:EB,$$

it is also

$$DG:GH = GH:HE = DB:GB = GB:HB = HB:EB$$

Because it is

$$DB:GB = GB:HB,$$

by subtracting you have

$$\begin{aligned} DB:(DB - GB) &= GB:(GB - HB) \\ DB:GB &= DG:GH \end{aligned}$$

The proportion involving GH and HE is obtained in the same manner. With a further reasoning on the proportions, it is possible to prove that it is also:

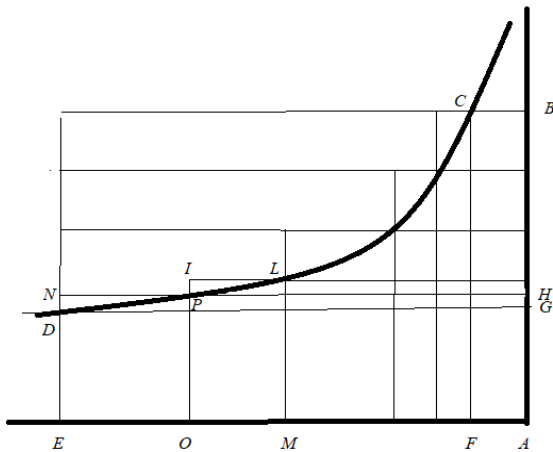
$$AB:BC = DG:GH = GH:HE = DB:GB = GB:HB = HB:EB$$

Since $AB:BC = HB:EB$, the rectangles of diagonals AE and HC are equal. Thus, the rectangles AN and HN are also equal. Basing on this fundamental acquisition, with another argument, Torricelli demonstrates that the ratio between the rectangles DN to AN is greater than 3 (Torricelli wrote "4", but we are confident that he meant "3"). The proves concerning the items 2) and 3) of the fundamental Lemma (Cfr. Bortolotti 1925b, p. 946) are obtained by other arguments which also include the use of barycentres.

Torricelli analyses some other specific cases, for example $m=3, n=7$.

Bortolotti proved the lemma through algebra. His demonstrations cannot be ascribed to Torricelli because they belong to an order of ideas which is not that of Torricelli, who reasoned through geometry.

The next step of Torricelli's analysis of infinite hyperbolas concerns the quadratures (Fig. 53):



Consider the hyperbola CD . Given an orthogonal system of coordinates with origin A where the asymptotes AE and AB of the hyperbola mutually intersect, be (F, B) and (E, G) the coordinates of C and D ⁶³. The mixtilinear quadrilateral $EDCF$ is to the mixtilinear quadrilateral $DCBG$ as the exponent n of BA is to the exponent m of EA .

Fig. 53. Our representation of the figure used by Torricelli to explain the reasoning above argued.
(Torricelli 1919-1944, I, 2, p. 256)

Torricelli developed a reasoning by exhaustion in the following manner:

Suppose that $\frac{EDCF}{DCBG} < \frac{n}{m}$ and that $\frac{EDCF}{D'C'B'G'} = \frac{n}{m}$.

Bisect continuously the rectangle $DLBG$ till reaching a rectangle DH such that its area is less than $DCBG - D'C'B'G'$.

Inscribe in $DCBG$ a figure composed of rectangles as high as DH . They are the rectangles IG, LH , etc.

⁶³ Torricelli does not use this language typical of analytical geometry, but, in this case, a more modern language does not betray his thought. It must be pointed out that the diagram (Torricelli 1919-1944, I, 2, p. 256) has many defects. Instead of the letter F you find P . The letter I is not indicated. We have not seen the original diagram in the manuscript, but at least a note by the editors would have been necessary to explain the defects of such a figure. In our adapted diagram we edited it.

Therefore, the inequality $\frac{EDCF}{\sum \text{rectangles } IG, LH, \dots} < \frac{n}{m}$ also holds.

However, this is absurd. As a matter of fact, $\frac{DO}{IG} > \frac{n}{m}$, because of the fundamental Lemma (Bortolotti 1925b, p. 946).

Thus, the quadrilinear $EDIO$ is such that $\frac{EDIO}{IG} > \frac{n}{m}$. The same reasoning holds for the rectangles PM and LH and for all the inscribed rectangles.

Therefore, the whole quadrilinear $EDCF$, which is bigger than the sum of the rectangles DO , IM , etc has a ratio greater than n/m in respect to the sum of the rectangles IL , LH , ..., which is absurd.

By means of an analogous reasoning, Torricelli also proved to be absurd the hypothesis

$$\frac{EDCF}{DCBG} > \frac{n}{m}$$

One of the elements in this ingenious proof – as it is typical of many reasonings by exhaustion – consists in regarding a mixtilinear figure as composed of rectilinear figures which approximate it better the more numerous they are. The idea behind these reasonings is that they are valid whatever the number of composing figures. One can say that they are valid even in the case that their number tends to infinity.

This is what Torricelli proved.

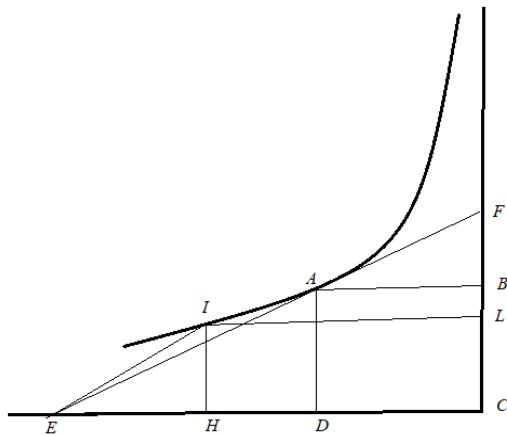
Bortolotti (Bortolotti 1925ab; see also 1928b, 1939) started from Torricelli's result that

$$\frac{EDCF}{DCBG} = \frac{n}{m}$$

and developed a series of reasonings, which are not present in Torricelli's work, to reach the conclusion that

$$\int_{x_1}^{x_2} x^{\frac{m}{n}} dx = \frac{1}{\frac{m}{n} + 1} \left(x_2^{\frac{m}{n} + 1} - x_1^{\frac{m}{n} + 1} \right)$$

Bortolotti does not claim explicitly that Torricelli achieved this result. But it is rather obvious that this is his idea. Through the fundamental Lemma (Bortolotti 1925b, p. 946) Torricelli is also able to determine the tangents at any point of a hyperbolic curve. His theorem states that (Fig. 54):



Given a hyperbola of asymptotes EC and CF , it is required to find the tangent at A . Draw AB and AD resp. parallel to the asymptotes. Choose the point E such that $ED:DE = CB:BF = n:m$. The straight line EF is the required tangent.

Fig. 54. Our representation of the figure used by Torricelli to explain the reasoning above argued.
(Torricelli 1919-1944, I, 2, p. 257)

Bortolotti (Bortolotti 1925b) starting from the fundamental lemma by Torricelli develops the following interesting reasoning, which also includes some propositions proved in the memoir *De infinitis parabolis* (Torricelli 1919-1944, I, 2, pp. 275-328).

The context concerns the way to draw the tangent to a parabola of the form:

$$y^n = kx^m$$

Torricelli considered the parabola ABC . The quadrilateral $BCFE$ is to the quadrilateral $BCGH$ as n to m . Bisect FE in I . Draw the abscissa ID and from D the ordinate DL . The ratio n/m also holds between the two quadrilaterals $DCEI$ and $DCGL$. Now Torricelli claims:

If we will do this division an infinite number of times or if we suppose that such a division is done, two lines CE and CG will remain instead of the two figures [the quadrilaterals]. These lines will be in the same ratio as the exponents, not according to their length [longitudine] but according to their quantity.⁶⁴

Bortolotti developed (Bortolotti 1925ab) the following argumentation: if you consider AE and AG as the coordinates of a point namely (to use Torricelli's language, if you consider them in length) the equation $AE^n = kAG^m$ subsists. If you regard them as indivisibles, because of the conservation of the formal properties, the ratio n/m holds. Therefore, according to Torricelli, the indivisibles are composed of two factors: a finite factor, which is the ordinate or the abscissa of a point belonging to the curve and an infinitesimal factor. For the ordinate such a factor is the thickness of the abscissa and for the abscissa the thickness of the ordinate. Thus, these indivisibles would be written in modern symbolism as ydx and xdy . Hence, these are elements of areas, operating with which it is possible to find the tangents to the hyperbolas or parabolas. Therefore, the relations between the elements of area can be written as

$$\frac{ydx}{xdy} = \frac{n}{m}$$

According to Bortolotti (Bortolotti 1925ab; 1928b), Torricelli arrived very close to the concept of integral and of differential. Both Torricelli's use of exhaustion and of indivisibles can be interpreted as fundamental steps towards infinitesimal calculus⁶⁵. Especially the previous relation is considered fundamental by Bortolotti because it reduces all the properties of a geometrical figure to a unique principle. This represented a novelty, which influenced Barrow and was at the basis of Leibniz's concept of differential (Bortolotti 1939, pp. 466-467; Pisano, Fichant, Bussotti and Oliveira 2017; Bussotti and Pisano 2017; Bussotti 2015). Thus, Bortolotti claim that:

- a) Though Torricelli does not speak of differential triangle, he supposes implicitly its existence (Bortolotti 1939, p. 471);
- b) Torricelli's concept of definite integral was absolutely clear (Bortolotti 1939, p. 471-472).

⁶⁴ "Et se faremo, o supporremo fatta questa divisione infinite volte, resteranno in cambio di figure due linee CE , et CF [CG], le quali, non secondo la longitudine, ma secondo la quantità saranno nel medesimo modo come l'esponente all'esponente." (Torricelli 1919-1944, I, 2, p. 322; Authors' translation). We point out that there is a mistake: sometimes you find letter F instead of the letter G in the Torricelli's text. But it is obvious when G is necessary. The editors do not note this problem. In the translation, we have reported the correct letter. In the original Italian text both those correct and those incorrect.

⁶⁵ Bortolotti also addressed questions such as the fundamental theorem of calculus and indefinite integrals in Torricelli, but what we have expounded is sufficient to give a clear idea of his opinions. Bortolotti expressed the idea that Torricelli's conceptions are close to those of the inventors of infinitesimal calculus in several works. Although we do not share his opinion, his contribution is probably the most complete and informative on Torricelli (Bortolotti 1921, 1922, 1923, 1924, 1925ab, 1928a, 1928b, 1931, 1937, 1939, 1947). An author who wrote important contributions on Torricelli is Amedeo Agostini. Basically, he followed Bortolotti's train of thought. (Agostini 1930a, 1930b, 1930c, 1931, 1949, 1950, 1951a, 1951b, 1951c, 1951d, 1951e).

5. Infinitesimals and Indivisibles

In this section, we do not propose addressing all the nuances of the concept of the indivisible. This would be too broad a perspective for our Introduction. Rather we propose these ideas focusing on Torricelli:

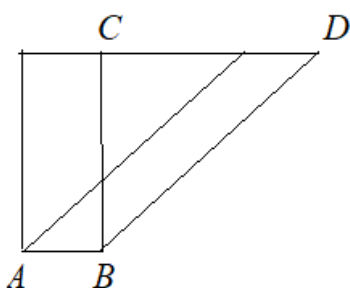
- 1) From a conceptual point of view, indivisibles represent, so to speak, “a third road” to reach results concerning curvilinear or mixtilinear figures, as well as to solve physical problems. The two others are the exhaustion method and infinitesimal calculus.
- 2) The concept of limit is not involved in indivisibles. This is what prevents us seeing in the indivisible technique a precursor of infinitesimal calculus. Obviously, this does not exclude that the inventors of infinitesimal calculus were inspired by Cavalieri and Torricelli, nor that the results expressed through indivisibles can be transformed into the language of differential and integral calculus.
- 3) Torricelli’s indivisibles are different from Cavalieri’s. In the former, the concept of actual infinitesimal is involved, whereas in Cavalieri this is not the case.
- 4) The method closer to indivisibles is that of Archimedes’s barycentres.

An interesting account of Torricelli’s ideas on indivisibles can be found in *De infinitis parabolis*. The following explanations concern the section entitled “Delle tangenti delle parabole infinite per lineas supplementares” (“On the tangents to the infinite parabolas through supplementary lines”; Torricelli 1919-1944, I, 2, pp. 320-323).

Torricelli claimed it to be false that all the indivisibles of the same kind are equal. Not all the points are equal. The lines have a width. The width of different lines is different. The surfaces have a thickness. The thickness of different surfaces is different. Torricelli’s examples are clear: consider two concentric circles and, from the centre, draw a series of straight lines cutting both the external and the internal circumference. There is a bi-univocal correspondence between the points cut by such lines in the internal and in the external circumference. However, these points are not equal: be p_e, p_i resp. points belonging to the external and the internal circumference and d_e, d_i the respective diameters. It is:

$$p_e : p_i = d_e : d_i$$

Consider another significant example given by Torricelli as presented in the following figure (Fig. 55):



Be given two parallelograms on the same base AB . In both of them, draw the infinite lines parallel to the sides (“infinite linee parallele ai lati” (Torricelli 1919-1944, I, 2, p. 321). The aggregate of the straight lines parallel to BC will be equal to that of the straight lines parallel to BD because the parallelograms are equivalent. Their number is also equal. What is different is their respective length and width. Those in the parallelogram AC are wider and shorter than those in the parallelogram AD .

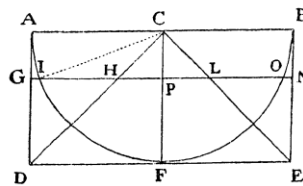
Fig. 55. On two parallelograms and infinite lines parallel to the sides
(Torricelli 1919-1944, I, 2, p. 321)
Source: Public domain

Hence, it seems to us that no doubt can subsist: according to Torricelli, indivisibles are actually infinitesimal quantities, which compose the figures. The figures are the sum of their indivisibles. The indivisible points, lines and surfaces are not the Euclidean points, lines and surfaces. An infinite aggregate of Euclidean points does not compose a surface, whereas an infinite aggregate of indivisible points does. They are actual infinitesimals whose number is infinite in any finite segment. The same reasoning can be extended to indivisible lines (be they straight or curved) and surfaces.

Torricelli continues with a further interesting observation: Galileo was right while claiming that a point can be equal to a line. He is referring to the famous bowl of Galileo (Fig. 56):

SAGR. Ammirabil proposta veramente mi par cotesta: però sentiamone l'esplicazione e la dimostrazione.

SALV. È necessario farne la figura, perchè la prova è pura geometrica. Per tanto intendasi il mezzo cerchio AFB, il cui centro C, ed intorno ad esso il parallelogrammo rettangolo ADEB, e dal centro a i punti D, E siano tirate le rette linee CD, CE; figurandoci poi il semidiametro CF, perpendicolare a una delle due AB, DE, immobile, intendiamo intorno a quello girarsi tutta questa figura: è manifesto che dal rettangolo ADEB verrà descritto un cilindro, dal semicir-



colo AFB una mezza sfera, e dal triangolo CDE un cono. Inteso questo, voglio che ci immaginiamo esser levato via l'emisferio, lasciando però il cono e quello che rimarrà del cilindro, il quale, dalla figura che riterrà simile a una scodella, chiameremo pure scodella: della quale e del cono prima dimostreremo che sono eguali; e poi, un piano tirato parallelo al cerchio che è base della scodella, il cui diametro è la linea DE e centro F, dimostreremo, tal piano, che passerà, v. g., per la linea GN, segnando la scodella ne i punti G, I, O, N, ed il cono ne' punti H, L, tagliare la parte del cono CHL, eguale sempre alla parte della scodella, il cui profilo ci rappresentano i triangoli GAI, BON; e di più si proverà, la base ancora del medesimo cono, cioè il cerchio il cui diametro HL, esser eguale a quella circular superficie che è base della parte della scodella, che è come se dicessimo un nastro di larghezza quanta è la linea GI (notate intanto che cosa sono le definizioni de i matematici, che sono una imposizion di nomi, o vogliam dire abbreviazioni di parlare, ordinate ed introdotte per levar lo stento tedioso che voi ed io sentiamo di presente per non aver convenuto insieme di chiamar, v. g., questa superficie, *nastro circolare*, e quel solido acutissimo della scodella *rasoio rotondo*): or comunque vi piaccia chiamargli, bastivi intendere che il piano prodotto per qualsivoglia distanza, pur che sia parallelo alla

Be given the cylinder. Inscribe the cone of vertex C. Consider the hemisphere inscribed in the cylinder. Subtract the hemisphere from the cylinder. The remaining solid was called by Galileo "bowl" ("scodella"). He proved that the volume of the bowl is the same as that of the inscribed cone.

1. cotesta: e però — 5. parallelogrammo rettangolo, s — 7. semidiametro CF immobile — 9. rettangolo, s — 21. per la linea GI, segnando, s — 27. di larghezza quanto è — 34. In G era stato scritto parallelo, che GALILEO corresse di sua mano in parallelo. —

Fig. 56. A representation of Galileo's "scodella" (bowl)

(Galilei 1890–1909, VIII, Day I, p. 74)

Source: French National Library – Gallica. Public domain

Galileo proved the proposition through indivisibles, demonstrating that each slide s_1 of the bowl has the same surface as the slide s of the cone. If the plane α cutting the figure moves from b_1 to b_2 , in the bowl the circumference c remains, while in the cone the point O . Galileo claimed:

Now, since as these solids diminish equality is maintained between them up to the very last, we are justified in saying that, at the extreme and final end of this diminution, they are still equal and that one is not infinitely greater than the other. It appears therefore that we may equate the circumference of a large circle to a single point.⁶⁶

⁶⁶ "Or mentre che nella diminuzione de i due solidi si va, sino all'ultimo, mantenendo sempre tra essi la egualità, ben par conveniente il dire che gli altissimi ed ultimi termini di tali menomamenti restino tra di loro eguali, e non l'uno infinitamente maggior dell'altro: par dunque che la circonferenza di un cerchio immenso possa chiamarsi eguale a un sol punto." (Galilei 1890–1909, VIII, Day I, p. 75; Translation from Galilei 1954, p. 28).

Galileo seems to be astonished at this conclusion, possibly because he neither has an exact concept of limit nor an idea like Torricelli's that indivisibles entities are different from the corresponding Euclidean entities. Torricelli, instead, is not surprised, at all. We read:

What Galileo claims with regard to a point equal to a line is true. In our hyperbolic solid it is true that the infinitely long axis is equal to a cylindric surface, or to a given [quanto] circle.⁶⁷

Torricelli is arguing that the indivisible given by the infinitely long axis of the solid is regarded as an infinitely long cylinder whose base circumference has an infinitesimal radius; infinitesimal, but not null. Multiplying the infinitesimal base circumference by an infinitely long height, it is possible to obtain a finite quantity equal to the other cylinders, which compose the hyperbolic solid. Thus, the length of the axis is an infinite *given* quantity and the length of the base circumference is an infinitesimal *given* quantity. There is no limit process. This is not only a theoretical assumption. As a matter of fact, the proofs by infinitesimals we have analysed do not involve considerations based on the concept of limit, which is, instead, the fundamental one in infinitesimal calculus. Bortolotti (Bortolotti 1939), as to the axis of the hyperbolic solid writes:

At limit, when $x=0$, the cylindric surface [involucro] degenerates in the axis of the conoid. Its length is infinite and its radius is null. However, because of the formal properties' conservation, we can consider, at the limit, too, the area of a cylindric surface [involucro] equal to that of the circle OD [one of the circles composing the cylinder at which the hyperbolic solid is equal]. As a matter of fact, the circles of the cylinder $OAED$ [the circle at which the solid is equal] are to be regarded as discs of infinitesimal thickness. The cylindrical surfaces [involucri], as well, are not Euclidean surfaces, but have an infinitesimal thickness.⁶⁸

By using the modern concept of limit, the identity between the volume of the hyperbolic solid V_{hs} and the cylinder's V_c means:

$$\forall \varepsilon > 0, \exists \delta > 0 / |x| < \delta \rightarrow |V_{hs} - V_c| < \varepsilon.$$

Therefore, the known volume of the cylinder is as good an approximation as you could desire of the hyperbolic solid's volume. There is no consideration regarding the axis of the solid. However, Torricelli does not reason in this manner. He considers the hyperbolic solid as composed of an infinity of hollow cylinders whose height is finite and whose thickness is infinitesimal. The finite cylinder of equal volume is instead composed of an infinity of cylinders whose base surfaces are finite and whose height is infinitesimal. Therefore, if the first paragraph of Bortolotti's quotation (see above; Bortolotti 1939) is taken as referring to the modern concept of limit, it is, at best, obscure. If it is taken as interpreting as the scientific cognitive process which led Torricelli to his result, it is possibly correct. With regard to the second paragraph of the previous passage, if "infinitesimal" means "actual infinitesimal", the idea expressed by Bortolotti (*Ibidem*) is agreeable. But this seems in contradiction to the whole interpretation advocated by Bortolotti (*Ibidem*). Hence, by the term "infinitesimal" he means "potentially infinitesimal". In contrast to this, we have argued that Torricelli's conception has nothing to do with the notion of limit. An actually infinite number of objects whose magnitude is actually infinitesimal is at stake. Thus, according to our view, Torricelli considers the figures as sums of infinitesimal magnitudes of the same kind. Torricelli's segments are sums of extended points, which, in fact, are infinitesimal segments. *Mutatis mutandis*, the same considerations hold for

⁶⁷ "Quello che dice il Galileo di un punto eguale ad una linea è vero e nel nostro solido hyperbolico è vero che l'asse infinitamente lungo è eguale ad una superficie cilindrica, ovvero ad un circolo quanto." (Torricelli 1919-1944, I, 2, p. 321: Authors' translation).

⁶⁸ "Al limite per $x=0$, l'involucro cilindrico degenera nell'asse del conoide, la sua lunghezza è infinita ed il raggio nullo. Ma per la conservazione delle proprietà formali, possiamo considerare, anche al limite, uguale l'area di un involucro cilindrico a quella del cerchio OD . Nel fatto poi, i cerchi del cilindro $OAED$, sono da considerarsi come dischi di spessore infinitesimo, ed anche gli involucri cilindrici, non sono superfici euclidee, ma hanno spessore infinitesimo." (Bortolotti 1939, p. 470).

Torricelli's surfaces and solids. One can say that Torricelli developed, in part, a calculus with actually infinite and infinitesimal quantities: his indivisibles. It is a different road both from the exhaustion method and from infinitesimal calculus. While referring to the secondary literature, Mancosu and Vailati expressed an opinion similar to ours:

[...] he [Torricelli] conceived of curved indivisibles as having a thickness; Cavalieri's indivisibles have a lower dimension than the figure they characterize, and therefore one cannot speak of a thickness of indivisibles in his account. Finally, Torricelli substituted for the complexity of Cavalieri's approach (in which the geometric figure and the class of indivisibles were considered as two separate magnitudes) a simple identification of the figure with its characterizing indivisibles (Mancosu and Vailati 1991, p. 53).

The two authors continue analysing Torricelli's statement and proof of the theorem on the acute hyperbolic solid. They express it like this:

The theorem as it stands, is about the proportion between two figures; an infinitely long solid and a cylinder. Thus the notion of actually infinite length is present in the very statement of the theorem. Moreover, the proofs themselves, as they stand, make sense only if the hyperbolic solid is given as infinitely long *in actu*. For we are comparing the volume of a fixed determinate solid with that of a cylinder, and there is no indication that the volume of the hyperbolic solid is thought of as the limit of a succession of volumes converging to that of the cylinder (Mancosu and Vailati 1991, p. 57).

The same opinion is expressed by Alexander who writes:

In fact, there were important differences between Cavalieri's and Torricelli's approaches to infinitesimals. Most critically, in Torricelli's method all the indivisible lines taken together really did make up the surface of a figure, and all the indivisible planes actually composed the volume of a solid. Cavalieri, as will be recalled, worked hard to avoid this identification, speaking of "all the lines" as if they were different from a plane and of "all the planes" as if they were different from a solid. But Torricelli had no such qualms. In his proof, he moves directly from "all the lines" to "the area itself" and from "all the planes" to "the volume itself", without bothering with the logical niceties that so concerned the elder. (Alexander 2013, p. 108).

Bascelli highlights that Torricelli's points are not indistinguishable entities, and neither are his linear indivisibles. Furthermore, Torricelli's indivisibles have the same dimension as the continuum they compose. They do not have a dimension less. After these agreeable considerations Bascelli claims:

Hence, the geometry of indivisibles is the usual plane geometry applied to an infinitesimal scale where dimensions are minute and finite (Bascelli 2015, pp. 127-128).

It is not possible that the dimensions of Torricelli's indivisible are finite. They are actually infinitesimal. As Bascelli herself admits (Bascelli 2015 pp. 112-113) and as it is absolutely evident from Torricelli's work, a figure is composed of an infinite number of indivisibles. This would be impossible if their dimensions were finite. They have to be actually infinitesimal. Obviously admitting actually infinitesimal quantities implies several problems, but this notion is crucial for Torricelli's geometry (Cfr. De Gandt 1987, 1992).

Torricelli had a less problematic attitude than Cavalieri towards indivisibles. The latter addressed many logical nuances of indivisibles. He wrote with regard to their nature:

I have used an artifice [the indivisibles] similar to that used by the algebraists to solve the given problems. Actually, though being the roots ineffable, absurd, and unknown, they add, subtract, multiply and divide them. They are convinced to have done their duty provided that they find the required information from the given problem. Thus, in the same manner, I have resorted to the multiplicity of the indivisibles (be they lines or planes) to investigate the measure of the continuous quantities. [...] This

has been the case, though their number is unmentionable, absurd and unknown. It is, however, included in visible limits as to its size.⁶⁹

Therefore, Cavalieri is claiming that his indivisibles are not necessarily interpretable as real mathematical entities, which compose the continuum. They can be regarded as artifices useful to reach a certain purpose. They are subject to precise rules. For example, it is necessary to distinguish “retto transito” and “obliquo transito”⁷⁰. However, they have the same status as the imaginary numbers used to solve the equations. They are a part of the symbolism. This means that the continuum is not composed of indivisibles, but it can be studied through them. Actually, it is not clear to grasp exactly what Cavalieri thought in regards to his indivisibles. It is certain that he assumed the quasi-operationalist position we have referred to, but what were the indivisibles from an ontological point of view, according to Cavalieri? The historians of mathematics have expressed different opinions: Guido Castelnuovo (1865–1952) suggested that, according to Cavalieri, a flat, finite area is composed of the infinite numbers of chords intercepted by a system of parallel straight lines. Each of these chords is regarded as an infinitesimal rectangle, an idea which we ascribe to Torricelli’s indivisibles rather than to Cavalieri’s. However, Castelnuovo continued, the *Jesuat scientist* denied resorting to actual infinitesimals and ascribes the interpretative difficulties involved in indivisibles to the ambiguities of Cavalieri’s language. Castelnuovo claims that Cavalieri made no mistake, but that his indivisibles cannot be regarded as an anticipation of the concept of definite integral because there is no reference to the concept of limit, a step which was taken by Pietro Mengoli (1626-1686) thirty years before Newton (Cfr. Castelnuovo 1962, pp. 44-49). Lucio Lombardo Radice (1916–1982) in his interesting Introduction to Cavalieri 1966, claims two theses:

- a) The rationale behind the method of indivisibles is essentially different from that behind integral calculus (Lombardo Radice in Cavalieri 1966, p. 10);
- b) The indivisibles are the first example in which the mathematics of infinity is developed.

Cavalieri laid the initial bases of operations with infinite sets: he did not think that a continuum of degree n is the sum of an infinite aggregate of continua whose degree is $n-1$. However, the points of a line, the lines of a flat region and the planes of a solid are perfectly conceivable. Each of these three aggregates is composed of an infinite number of elements (Lombardo Radice in Cavalieri 1966, pp. 24-26 and pp. 652-653). Andersen, Giusti and Jullien avoided both the theses that Cavalieri was a precursor of integral calculus and of infinite sets (Andersen, Giusti and Jullien 2015; Jullien 2015). There is no sum, as is the case with integral calculus, in Cavalieri’s method. On the other hand, Cavalieri’s method seems directly connected to the measure of a figure, which can hardly be obtained in a set-theoretical context without distorting some of Cavalieri’s basic assumption. Thus, indivisibles enter the general – and admittedly not always transparent – notion of magnitude. After a refined analysis of the notion of the indivisible, the three authors attempt to draw a conclusion: although Cavalieri avoided speaking of the sum of indivisibles and even if most of his theorems do not require such an assumption, there are some basic propositions in which the idea that a figure is composed of the sum of its indivisibles seem necessary (Andersen, Giusti and Jullien 2015, p. 55; Jullien 2015). The legacy of Cavalieri’s indivisibles had already been given by Giusti and Andersen in their contributions published resp. in 1980 and 1985. Here the procedures of the Jesuat mathematician are transcribed into a clear calculus; the differences with integral calculus are pointed out; the collective and distributive method is clearly explained. Andersen also clarifies that, among the European

⁶⁹ “*Artificio autem tali sum, quale ad propositas quaestiones ab solvendas Algebraici adhibere solent; qui quidem numerorum radices, quamvis ineffabiles surdas, ac ignotas, nihilominus simul aggregantes subtrahentes, multiplicantes, dividentes, dummodo propositae rei exoptatum sibi notitia enucleare valeant, sua satis obijssse numera sibi persuadent, Non aliter ergo indivisibilium sive linearum, sive planarum congerie (ijsdem ut in lib. 2. explicatur assumptis) licet quoad eorundem numerum innominabilis, surda, ac ignota, quoad magnitudinem tamen conspicuis limitibus clausa, ad continuorum investigandam mensuram usus sum, ut legenti in processu operis apparebit.*” (Cavalieri 1635, *Prefatio*, p. 3 (s.n.); Cavalieri italic style. Authors’ translation).

⁷⁰ On the distinction between “retto transito” and “obliquo transito” see, i.e., Cavalieri’s letter to Galileo on 2 October 1634 (Galilei 1890-1909, XVI, p. 138). This letter is also presented by Cavalieri (Cavalieri 1966, pp. 754-757; see also Andersen 1985; pp. 308-310; Mancosu 1999, p. 236).

mathematicians, Torricelli's conception of indivisibles rather than Cavalieri's was spread (Cfr. Andersen 1985, pp. 353-363).⁷¹

Torricelli's attitude is more realistic and pragmatic: the indivisibles really compose the continuum. This does not mean that he was not interested in the logic behind indivisibles' correct use. He wrote, e.g., a text in which a series of incorrect utilizations of indivisibles are presented (Torricelli 1919-1944, I, 2, pp. 415-432). For each example, he developed a set of logical and mathematical considerations aimed at revealing the mistakes hidden in the reasoning. But undoubtedly he had a realistic way of conceiving the indivisibles.

Finally, as we have shown, it must be noted that Torricelli's conception of indivisibles and of infinite and infinitesimal quantities is rather different from that of the inventors of infinitesimal calculus. In the same period in which Torricelli conceived the indivisibles, the author who had ideas closer to those connoting calculus was surely Fermat (Bussotti and Pisano 2020). This does not exclude that, as already mentioned, the inventors of calculus were inspired by Cavalieri's and Torricelli's ideas. We do not deal with this subject because the literature is relatively abundant. We prefer to stress that a concept, which seems in filiation with the notion of indivisible exists in physics. It is that of the "minimal lines of given length" or "minimal surfaces of given thickness".

In the Proposition XXXIX of section VII, Book I of *Principia*'s first book Newton solved the following problem:

Supposing a centripetal force of any kind, and granting the quadratures of the curvilinear figures; it is required to find the velocity of a body, ascending or descending in a right line, in the several places through which it passes; as also the time in which it will arrive at any place; And vice versa.⁷²

⁷¹ With regard to the spread of indivisibles the whole collective book Jullien 2015 is a precious document. From a mathematical perspective an interesting paper is Carroll, Dougherty and Perkins 2103, where the authors revisit in modern forms some techniques by Torricelli and Roberval, without betraying their original procedures. They also valuate how justified Guldin's criticisms were. Among the works dealing with Cavalieri's indivisibles we also mention De Gandt 1991, where a plurality of noteworthy questions are addressed. The author clarifies that the choice to admit the existence of indivisibles is, eventually, axiomatic; Festa 1992, where the author, among other themes, also faces the issue concerning the relation between a figure and its indivisibles; Palmieri 2009. This is a remarkable paper on the manner in which Cavalieri used the technique of superimposing figures.

⁷² Newton [1726]1729, p. 163.

In the course of the proof, Newton used this concept (Fig. 57):

In a phase of his reasoning, Newton wrote:

[122]

vis AD , & corpus A , tempore AD , cadendo describet spatium AC , inq; loco C acquiritur velocitatem CD . Demonstratur eodem modo ex Propositione X . quo Propositio $XXXII$. ex Propositione XI . demonstrata fuit. Q. E. D.

Corol. 1. Hinc æqualia sunt tempora quibus corpus unum de loco A cadendo provenit ad centrum S , & corpus aliud revolvens describit arcum quadrantalem ADE .

Corol. 2. Proinde æqualia sunt tempora omnia quibus corpora de locis quibuscumque ad usq; centrum cadunt. Nam revolventium tempora omnia periodica (per Corol. 3. Prop. IV.) æquantur.

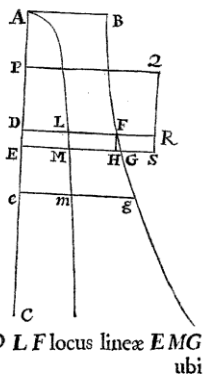
Prop. XXXIX. Prob. XXVII.

Posita cujuscumq; generis vi centripeta, & concessis figurarum curvilinearum quadraturis, requiritur corporis recta ascendens vel descendens tum velocitas in locis singulis, tum tempus quo corpus ad locum quemvis perveniet: Et contra.

De loco quovis A in recta $ADEC$ cadat corpus E , deq; loco ejus E erigatur semper perpendicularis EG , vi centripeta in loco

illo ad centrum C tendenti proportionalis: Sitq; BFG linea curva quam punctum G perpetuo tangit. Coincidat autem EG ipso motus initio cum perpendiculari AB , & erit corporis velocitas in loco quovis E ut area curvilinea $ABGE$ latus quadratum. Q. E. I. In EG capiatur EM lateri quadrato area $ABGE$ reciproce proportionalis, & sit ALM linea curva quam punctum L perpetuo tangit, & erit tempus quo corpus cadendo describit lineam AE ut area curvilinea $ALME$. Quod erat Inveniendum.

Etenim in recta AE capiatur linea quam minima DE datae longitudinis, sitq; DLF locus lineae EMG ubi



"In the straight line AE , be taken the as small as possible line DE of a given length and be DLF the place of the line EMG when the body was in D ."⁷³

Fig.57. The diagram used by Newton in Proposition XXXIX, Problem XXVII.

(Newton 1686, I, p. 122)

Source: French National Library – Gallica. Public domain

A line “quam minima datae longitudinis” cannot be regarded as a potentially infinite quantity. It seems an indivisible which composes the straight line AC and, from a formal point of view, it is. This does not mean that Newton believed his lineola DE to be an actual infinitesimal. He did not believe in the existence of actually infinitesimal quantities. The lineola does not belong to the ontology of mathematics. It is an artifice – which is not in contradiction with the rest of Newton’s concepts – to reach a certain purpose⁷⁴. In this respect, Newton’s “linea quam minima datae longitudinis” is closer to Cavalieri’s indivisibles than to Torricelli’s.

A comparison between the doctrine of indivisibles and Archimedes’s mechanical method can be developed. As we have seen, Torricelli thought that, behind Archimedes’s proofs by exhaustion, a heuristic method should exist. Heiberg’s discovery in 1906 was a spectacular confirmation of Torricelli’s ideas.⁷⁵ Furthermore, the mechanical method belongs to an order of thought close to that of

⁷³ “Etenim in recta AE capiatur linea quam minima DE datae longitudinis, sitque DLF locus lineae EMG , ubi corpus versabatur in D .” (Newton 1686, I, pp. 120-123; Authors’ translation).

⁷⁴ We have offered a broad treatment of the way in which Newton used this concept and others similar in Pisano and Bussotti 2017, 2014a, 2014b; Bussotti 2015, pp. 50-53. It is to point out that in physics until the 19th century it was rather common to resort to infinitesimal quantities whose ontological status was not very clear, but whose use was.

⁷⁵ It is probable that the mechanical method was a guide in Archimedes’s research. Therefore, under this respect, it has a heuristic value. However, as Acerbi stressed (Cfr. Archimedes 2018, p. 70; see also Archimedes

indivisibles. As a matter of fact, Archimedes argued the flat figure to be composed of straight lines and the spatial figure composed of surfaces. It is enough to take into account one passage in the first proof given by Archimedes within the argument to square parabola (Fig.58):

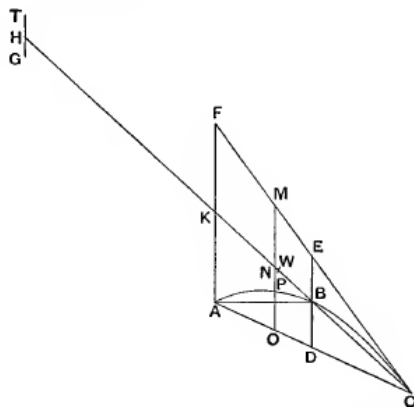


Fig. 53. “And, since the triangle CFA is made up of all the parallel lines like MO , and the segment CBA is made up of all the straight lines like PO within the curve [...]”. (Archimedes 1912, p. 17).

Fig. 58. The diagram used by Archimedes
(Archimedes 1912, p. 16)
Source: Public domain

Therefore, rectilinear figures are composed of an aggregate of straight lines. This idea is common to the doctrine of the indivisibles and to Archimedes' method. The concept to reach the solution of a problem resorting to the lever is a further fundamental feature shared by Archimedes' and some of Torricelli's procedures. Furthermore, the discoveries dating to the beginning of the 21st century of the palimpsest containing Archimedes' *Method* has shown us that Archimedes, at least in one circumstance – the cubature of the cylindrical hoof – used an authentic calculus with an actually infinite number of infinitesimal figures without resorting to the lever. This is close to what Torricelli did in many of the examples we have analysed. Therefore, there are several similarities between Torricelli's and Archimedes' methods. From an ontological point of view, it is very difficult to establish whether Archimedes thought that his indivisibles actually compose the figure or if they were an artifice to reach the required results. In other terms: Was Archimedes' view on the decomposition and composition of figures closer to Cavalieri's or Torricelli's? (Cfr. Pisano and Capecchi 2010a; Capecchi and Pisano 2007abc). The objects used by Archimedes are Euclidean entities, namely the lines have no width and the surfaces have no thickness. Since we exclude that Archimedes has fallen into such a serious mistake, we propend for the first hypothesis: Archimedes used his indivisibles as an artifice. He did not really believe that an infinite aggregate of continua of the order $n-1$ might compose a continuum of order n .

6. Conclusion

Torricelli is a significant example of the use the moderns made of the ancients' mathematical methods, rediscovering and employing their techniques, but adapting them to the needs of the new science.

In the first half of the 17th century some of the most skilled mathematicians tried to extend the results obtained by Archimedes to calculate the areas and the volumes of curvilinear and mixtilinear figures. They also aimed at transforming Archimedes' method of exhaustion into a more direct, malleable and general approach, which might allow them to overtake the results obtained by the great Syracusan. They were aware that to resort to infinitesimal methods in order to solve these problems was necessary. Therefore, the crucial question became how to develop a mathematics of infinite and infinitesimal quantities. At the beginning, we note Kepler's excellent works, but – for several aspects – his infinitesimal methods were unsatisfactory. Generally speaking, Bonaventura Cavalieri was one of the first mathematicians to offer a theory of infinitesimal quantities by means of his geometry of

1897), Archimedes presents its procedure in a highly formalized manner. Thus, Archimedes' presentation is not that typical of a heuristic procedure.

indivisibles. Setting aside some logical obscurities in Cavalieri's concepts, his results were of primary importance.

The main features of Cavalieri's method were:

- 1) The geometrical approach. His way of proceeding was still based on proportions rather than on equations. This aspect made his method certainly applicable to physics (statics) at that time, just before Newtonian mechanics.
- 2) The scarce use of algebra;
- 3) The lack of the concept of limit within indivisibles theory.

Evangelista Torricelli's contributions to geometry share the characteristics which connote Cavalieri's approach. Nonetheless, in Torricelli's work there is something more: he was both interested in solving the mathematical problems, and aimed at finding the tangible nature of the logical—mathematical structure behind geometry. According to his opinion, the exhaustion method, of which Torricelli was an expert and a *virtuoso*, was an excellent demonstrative procedure. However, beyond its lack of generality, it had a further problem: it did not grasp the nature of geometrical relations; it was extremely artificial. It proved the truth of a proposition, but did not go to the most profound root of geometry. In contrast to this, Torricelli was convinced that indivisibles reached the real grounds of the questions: they were a geometrical means through which the figures were decomposed into their atomic elements. For example, given two figures *A* and *B*, let us suppose that all the metric properties (i.e., perimeter, area, volume, depending on it being a flat or a solid figure) of the figure *A* and of its indivisibles are known. Whereas, the metric properties of the figure *B* have to be found, while we know some features of its indivisibles and of their relations with those of the figure *A*. Then, under enough general conditions, it is possible to deduce the metric properties of *B*.

According to Torricelli, this way of proceeding is purely geometrical, rigorous, direct and grasps the foundations connecting the different figures. It is not a coincidence that Torricelli considered figures as composed of their indivisibles and that gave a fundamental impulse to the development of this doctrine (as Archimedes did with his figures where the masses suspended to a lever moved jointly with it, also called *rigid movement*). For, he reached astonishing results such as those concerning the acute hyperbolic solid. Therefore, Torricelli was a mathematician and a researcher who contributed to the development of several theories within mathematical sciences. He did not reduce geometry to a series of techniques: for him, geometry had proper methods and the importance of using a correct method was as crucial as the result one reaches. As shown throughout the analysis, the diversity of procedures adopted by Torricelli emerges as a decisive element for understanding the originality of his results and for establishing his methodological independence. We think that this aspect is, perhaps, one of the reasons why Torricelli never cared about algebra. In fact, probably, for his aims, the use of algebra within geometry was an external and improper element. Without any doubt, this lack prevented him from fully appreciating the developments, which were taking place abroad, especially in France.

The results and the methods used by Torricelli in geometry suffice to ensure him a primary position within the scientific panorama of the 17th century. His extraordinary talent was not restricted to the branches of geometry and mathematics. Torricelli was a brilliant physicist, as well. He showed his talent as a polyhydry scientist also giving fundamental contributions to the study of the fluids and, as a matter of fact, he first conceived atmospheric pressure in the accurate manner. Besides his known studies on the barometer, in *Opera Geometrica*, he created (so-called nowadays) “Torricelli principle” in Mechanics (Pisano and Bussotti 2022) which is a crucial conceptual step in the theoretical physics of principles. It states that *two heavy bodies joined together cannot move by themselves unless their common centre of gravity descends* (Torricelli 1644, p. 99; *Id.*, 1919–1944, p. 105). The enunciation of Torricelli's principle actually arose from problems that were extraneous to the theory of the centre of gravity (Capecechi and Pisano 2007a). Specifically, it originated from the attempt to provide a more solid foundation for Galilean mechanics using this principle as a criterion of equilibrium in special cases only (Pisano and Bussotti 2025; Bussotti and Pisano 2025). By working both as an Archimedean and as a physicist, he dealt with the centre of gravity.

We briefly recall two fundamental steps developed by Torricelli in this theory:

- 1) The passage from the case study of the centre of gravity of a single body to the case study of that of a system of bodies;
- 2) The problem of convergence–parallelism with regard to the motions of heavy bodies falling toward the centre of the Earth.

Torricelli's idea was an analogical extension of the constraint imposed by the centrobaric theory, from the motion of the barycenter of a body to the motion of the barycenter of a system of rigid bodies. This allowed processing (before Newton's mechanics) a purely geometric point as if it were a point-like heavy body (in modern terms, *mass*). The Torricellian extension had both a qualitative nature and a quantitative one. It was supported by detailed geometrical–mathematical arguments. Particularly, his reasonings *ex suppositione* (Cfr. Torricelli 1644, pp. 9-15; Capecchi and Pisano 2010a) were logically consistent because they were confirmed by experience and results obtained by other *equilibrium criteria*, where the causes were investigated, too. He stated his principle as the foundation of dynamics and not of statics only. In fact, Torricelli enunciated his equilibrium principle for two bodies only, and then he extended it to situations of *unstable equilibrium* where the causes should have been discussed. To this end, more and less directly, he used infinitesimal displacements thereby also making clear the interplay between physics (mechanics) and mathematics (geometry). For this reason, his *equilibrium criterion* is substantially equivalent to the modern *principle of virtual works* (Pisano 2017) later on explicitly formulated in 1715 by Johann Bernoulli (1667–1748). Nevertheless and in any case, we remark that Torricelli enounced his principle before Christian Huygens's (1629–1695) and the English School of physicists, i.e., Isaac Barrow (1630–1677) and Robert Hook (1635–1703).

Opera Geometrica was on the borderline of the birth of modern science, so its scientific state was still in progress to find general principles in order to solve a set of problems with uniform methods; this granted, the Torricelli principle must be appreciated even more.

Finally, in order to inaugurate the celebration of the 380th *Opera Geometrica* anniversary (1644-2024), we summarized a selection of Torricelli's most important studies and discoveries. Many others might be added, as, e.g, the invention of a working set square, his further mathematical results concerning the logarithmic spiral. Several of these achievements were obtained in his outstanding *Opera Geometrica*. We also discussed the most remarkable results obtained by Torricelli after the publication of his *Opera Geometrica*. These results remained unpublished during Torricelli's lifetime. They show that Torricelli was further refining his mathematical technique and making his views broader.

Last but not least, we also underline Torricelli's studies—and—collaboration with Galileo at the end of his life, particularly during the pre-print of *Discorsi and dimostrazioni matematiche intorno a due nuove scienze*.

In conclusion this extensive essay, dedicated to Torricelli's *Opera Geometrica* anniversary (1644–2024) – within a celebrative huge volume on *Opera Geometrica* –, introduces the magnificent major work in Mechanics, Mathematics and Geometry of a brilliant Archimedean—and—Galilean scientist to the modern readers. Evangelista unfortunately died at the age of only 39, in the middle of his most productive period as a physicist, geometer and mathematician.

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Ritorna all'indice

La parabola di sicurezza di Torricelli

Fabio Ortolani*

Sommario. L'articolo esamina il concetto di parabola di sicurezza di Torricelli, pilastro della fisica classica e della balistica, illustrando sia il suo contesto storico sia la sua rilevanza teorica e applicata. Dopo una panoramica sul contributo di Torricelli e sull'evoluzione del concetto nel XVII secolo, viene descritta un'esperienza sperimentale condotta con un lanciatore di una biglia metallica, finalizzata a verificare empiricamente la validità delle leggi del moto parabolico. Attraverso l'utilizzo di strumenti di ripresa e software di analisi video sono state raccolte e confrontate le traiettorie a diversi angoli di ripresa, analizzando parametri come angolo, velocità iniziale e gittata massima. Viene approfondito il metodo matematico per determinare l'involuppo delle curve, ovvero la parabola di sicurezza, e vengono discusse le principali fonti di errore sperimentale e le strategie per ridurre l'impatto. I risultati mostrano una sostanziale concordanza tra teoria e pratica, confermano la validità dei modelli classici e offrono una riflessione sull'importanza della metodologia rigorosa nella didattica e nella ricerca scientifica.

1 Introduzione



Fig. 1 Torricelli

La parabola di sicurezza di Torricelli costituisce un principio fondamentale della fisica applicata, in particolare nell'ambito della balistica, delineando il limite massimo teorico raggiungibile da un proiettile lanciato con una specifica velocità iniziale. Tale concetto prende il nome da Evangelista Torricelli, eminente fisico e matematico italiano del XVII secolo, noto principalmente per i suoi studi sulla pressione atmosferica e come discepolo di Galileo Galilei.

Evangelista Torricelli (Roma, 15 ottobre 1608 – Firenze, 25 ottobre 1647), successore di Galileo presso la corte dei Medici a Firenze, fu tra i pionieri dell'analisi approfondita del moto dei proiettili. Nel suo trattato *De Motu Gravium* [1] egli approfondì lo studio delle traiettorie dei corpi in movimento, sviluppando ulteriormente le intuizioni già elaborate da Galileo nelle opere *Dialogo sopra i due massimi sistemi del mondo* [2] e *Discorsi e dimostrazioni matematiche* [3].

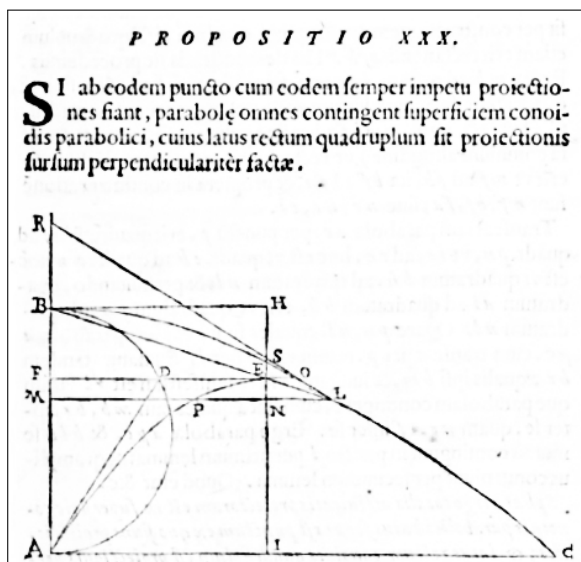


Fig. 2 30^a proposizione di Torricelli

Il XVII secolo rappresentò un periodo di straordinario sviluppo scientifico in Europa, caratterizzato dall'affermazione del metodo sperimentale e da un rinnovato approccio matematico allo studio dei fenomeni naturali. In questo contesto Torricelli formalizzò il concetto di parabola di sicurezza riuscendo a calcolare l'involuppo delle traiettorie paraboliche prodotte da lanci effettuati alla medesima velocità iniziale ma con differenti angoli di proiezione.

Nella figura 2 è riprodotta la proposizione originale di Torricelli che recita “Se si tira da uno stesso punto e sempre con lo stesso impeto, tutte le parabole risultano tangenti alla superficie del conoide parabolico, il cui lato retto è quadruplo dell'altezza raggiunta con un tiro verticale all'insù” (si veda il volume *Homage to Evangelista Torricelli's Opera Geometrica 1644–2024* [4]).

Questa acquisizione concettuale si rivelò particolarmente rilevante nel campo dell'ingegneria militare dove l'ottimizzazione della balistica risultava essenziale per incrementare l'efficienza dell'artiglieria.

* Docente dell'Università di Bologna – Dipartimento di Fisica e Astronomia fino al 2016.

Esperto in simulazioni numeriche di sistemi quantistici a molti corpi.

e-mail: f.ortolani53@gmail.com.

Il contributo teorico di Torricelli esercitò inoltre una significativa influenza su studiosi, quali Christiaan Huygens [5], Isaac Newton [6] e Johann Bernoulli [7], che approfondirono lo studio del moto e delle leggi dinamiche nei decenni successivi.

In epoca moderna il termine “parabola di sicurezza” è stato progressivamente assimilato anche nel linguaggio comune per indicare un confine insuperabile. L’espressione “oltrepassare la parabola di sicurezza” viene talora utilizzata per descrivere azioni considerate impossibili o obiettivi irraggiungibili. Questa locuzione ha travalicato gli ambiti della fisica e della balistica, consolidandosi nel lessico e nell’immaginario collettivo italiani. “Oltrepassare la parabola di sicurezza” sottolinea la natura utopica di determinati intenti analogamente a modi di dire quali “puntare alla luna” o “superare il muro del suono” e arricchisce la lingua italiana con efficaci metafore di origine scientifica.

L’ampio ricorso a questa espressione nella cultura popolare testimonia sia l’importanza storica della scienza in Italia sia la forza con cui concetti tecnici possono trasformarsi in simboli condivisi. Non mancano riferimenti alla “parabola di sicurezza” in opere divulgative, nei media o perfino in contesti politici, dove spesso viene richiamata per indicare la soglia tra ciò che risulta fattibile e ciò che resta, almeno temporaneamente, fuori dalla portata umana.

2 Verifica sperimentale della curva di sicurezza



Fig. 3 Il lanciatore.

Presso la *Palestra della Scienza di Faenza* [8] è stato utilizzato un cannoncino a molla, il modello *Pasco ME-6825B Mini Launcher* [9], appositamente progettato per il lancio di una biglia metallica a diverse angolazioni, per condurre un’esperienza mirata all’analisi dell’influenza dell’angolo di lancio sulla traiettoria e sulla distanza percorsa dalla biglia con l’obiettivo di verificare empiricamente i principi che regolano la parabola di sicurezza.

Per garantire condizioni ottimali e uniformità nei dati raccolti il dispositivo di lancio (figura 3) è stato fissato saldamente a una delle estremità di un tavolo lungo circa 3 metri. Il tavolo, realizzato in materiale liscio e privo di ostacoli, ha offerto una superficie regolare che favorisce la caduta della biglia senza interferenze e ne facilita il recupero dopo ogni lancio.

La configurazione adottata ha consentito la registrazione dei lanci effettuati a differenti angolazioni

permettendo così di analizzare con precisione le varie traiettorie della biglia. L’esperimento si è articolato in tre fasi operative:

- Riprese dei lanci: ogni lancio è stato registrato mediante una telecamera, al fine di acquisire una documentazione visiva dettagliata del moto della biglia.
- Analisi dei fotogrammi: sono stati analizzati i singoli fotogrammi di ciascun lancio con lo scopo di determinare con esattezza le traiettorie descritte dalla biglia per ogni angolo di lancio.
- Valutazione della curva di sicurezza: attraverso il confronto tra le traiettorie osservate e i principi teorici è stata valutata la curva di sicurezza, verificando empiricamente il concetto di parabola di sicurezza.

Questa metodologia ha reso possibile una verifica diretta e rigorosa dei principi fisici relativi alla parabola di sicurezza consentendo un’analisi dettagliata sia dal punto di vista qualitativo che quantitativo delle traiettorie e delle distanze percorse dalla biglia.

2.1 Riprese dei lanci

Dopo alcune prove preliminari sono state effettuate le riprese dei lanci regolando gli angoli di inclinazione ai seguenti valori nominali:

$$\theta = \{35^\circ, 45^\circ, 55^\circ, 65^\circ, 75^\circ, 85^\circ\} \quad (1)$$

Per l'acquisizione dei video abbiamo utilizzato una fotocamera *Nikon Z6* impostata in modalità video manuale con un framerate di 120 *fps* (massimo consentito) producendo sei filmati in formato *Full HD*. Sia il tempo di posa (1/800 *s*) sia l'apertura del diaframma (*F/11*) sono mantenuti costanti per tutti i filmati, mentre la sensibilità ISO è stata lasciata alla regolazione automatica del dispositivo per adattarsi alle condizioni di luce presenti in ciascuna ripresa e garantire immagini sempre brillanti.

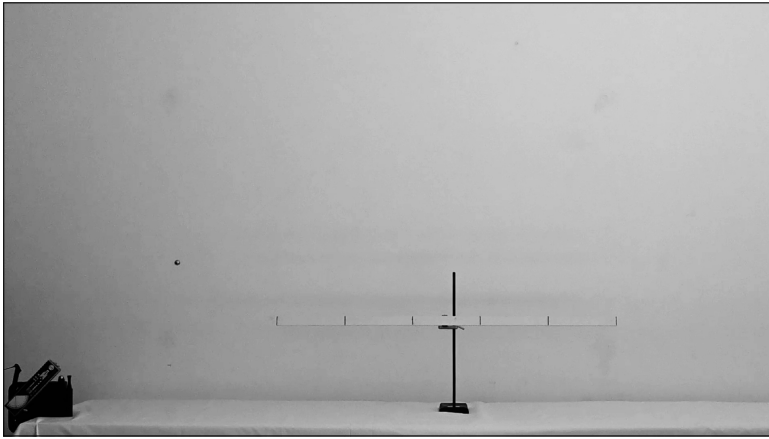


Fig. 4 Scena di ripresa

Questo approccio ci ha permesso di garantire coerenza nell'esposizione e nella profondità di campo tra i diversi lanci, limitando le variazioni alle sole condizioni di luminosità ambientale, al tempo stesso minimizzando le tracce di movimento della biglia e massimizzando la sua nitidezza. L'illuminazione della scena è stata fornita esclusivamente dalla luce naturale proveniente da due finestre in una giornata soleggiata e per tutte le riprese è stato impiegato un obiettivo *Nikkor Z 24-70 mm*. Un fotogramma del lancio a 45°

è mostrato in figura 4.

Successivamente i video sono stati elaborati tramite il software *LumaFusion* [11]: dopo una conversione in bianco e nero con lievi correzioni di luminosità e contrasto per avere immagini ancora più incisive, i filmati sono stati sincronizzati e allineati utilizzando come criterio il fotogramma esatto in cui la biglia lascia il lanciatore, così da ottenere inquadrature e angolazioni identiche e generare clip della stessa durata (1 *s*, corrispondenti a 4 o 5 *s* in time-lapse).

L'allineamento spaziale è stato effettuato sovrapponendo i punti di riferimento fissi sullo sfondo e sul tavolo, mentre la sincronizzazione temporale si è basata sull'analisi frame-by-frame del distacco della biglia dal lanciatore. Il momento esatto della partenza presenta un'incertezza intrinseca, poiché non è stato possibile coordinare in modo perfetto la registrazione della fotocamera con il rilascio della molla. Tale incertezza, pari a un intervallo massimo di 1/120 *s* (l'intervallo tra i singoli frame), è stata attentamente considerata nelle analisi successive includendo il margine di errore nelle valutazioni quantitative delle traiettorie e delle velocità iniziali per garantire la correttezza dei risultati.

2.2 Analisi dei video e raccolta dei dati

L'analisi sperimentale delle traiettorie della biglia è stata condotta utilizzando il software *Tracker* [10], uno strumento che consente di tracciare con precisione il moto di un oggetto in una sequenza video. Per ogni fotogramma dei video acquisiti è stata registrata la posizione della biglia lungo il suo percorso. Questa fase del lavoro ha richiesto particolare accuratezza poiché la determinazione esatta della posizione è fondamentale per la validità dei dati sperimentali e delle analisi connesse.

Il programma permette di fissare un sistema di riferimento e le unità di misura usate. Inizialmente, per calibrare le distanze nei video, si era pensato di utilizzare un'asta di riferimento lunga 1 metro. Questa soluzione si è dimostrata insoddisfacente a causa delle distorsioni prospettiche, introdotte dall'inquadratura, che avrebbero compromesso la precisione delle misurazioni. Per superare questo limite si è deciso di impiegare la biglia stessa come riferimento di scala sfruttando la sua dimensione nota di 16 *mm*. Sebbene identificare con esattezza la biglia in ogni fotogramma sia un'operazione complessa,

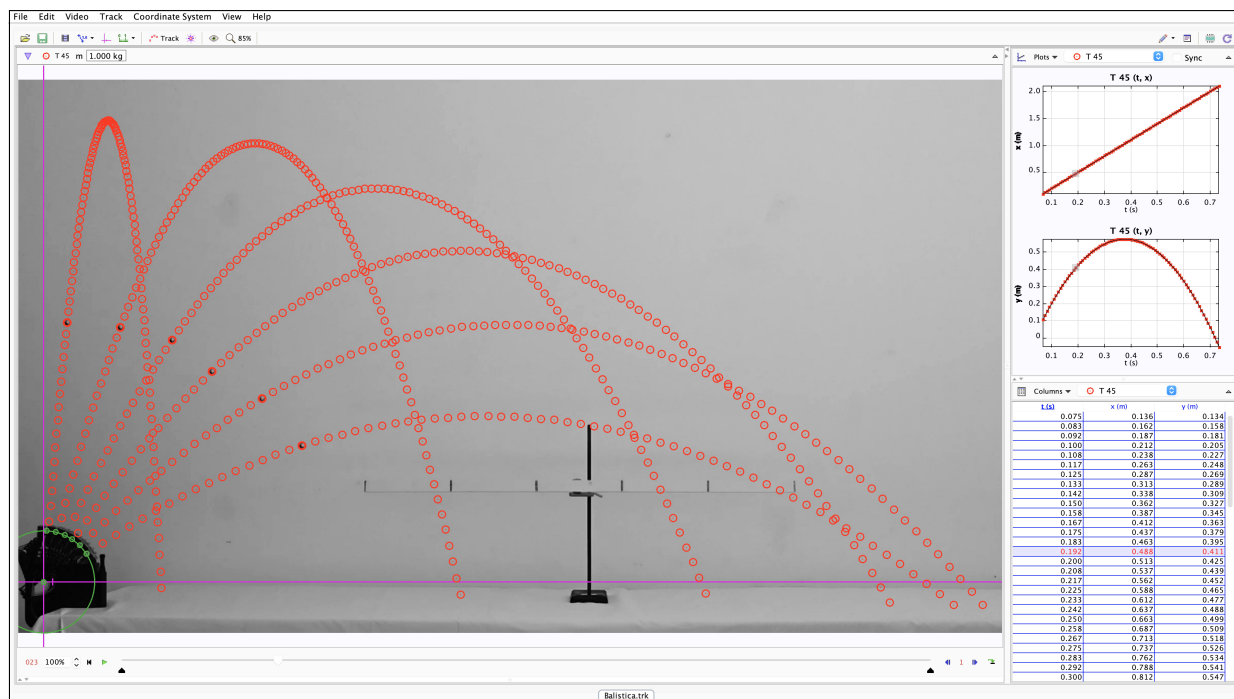


Fig. 5 Schermata del tracker con le tracce delle traiettorie.

a causa delle sue piccole dimensioni, questa scelta ha permesso di migliorare significativamente la precisione delle misure. È importante sottolineare che eventuali errori sistematici nella definizione della scala non influiscono sui rapporti tra le grandezze misurate, perché tutte le misure vengono scalate in modo uniforme e ciò mantiene invariati i rapporti e consente analisi comparative attendibili tra le differenti traiettorie osservate.

La figura 5 riassume graficamente il risultato di questa fase di lavoro mostrando le tracce sovrapposte delle traiettorie ottenute componendo i dati estratti dai diversi video. In questa rappresentazione sono stati individuati e segnati i punti corrispondenti alle posizioni iniziali di lancio che coincidono con il punto in cui, sul dispositivo, viene impresso l'impulso iniziale alla biglia. Al fine di rendere omogenee le misurazioni tutti i dati raccolti sono stati riferiti a una stessa origine determinata tramite una procedura di fitting e coincidente con il perno del cannoncino.

Il software *Tracker* si distingue per le sue funzionalità avanzate nell'analisi quantitativa delle traiettorie, consentendo non solo la registrazione accurata delle coordinate spaziali della biglia ma anche l'applicazione di sofisticati metodi di fitting matematico ai dati sperimentali raccolti. In particolare, nell'ambito di questo studio, sono stati utilizzati fitting polinomiali sulle coordinate temporali che hanno permesso di derivare le equazioni orarie $x = x(\theta, t)$ e $y = y(\theta, t)$ per ogni angolo di lancio θ

$$\begin{cases} x(\theta, t) = b_1 t + c_1 \\ y(\theta, t) = a_2 t^2 + b_2 t + c_2 \end{cases} \quad (2)$$

con i coefficienti polinomiali dipendenti da θ . Tali equazioni descrivono in modo rigoroso l'evoluzione spaziale della biglia in funzione del tempo e dell'angolo di lancio θ , offrendo una rappresentazione parametrica dettagliata del moto. In aggiunta, sono stati implementati fitting espliciti di tipo $y = y(\theta, x)$ tra le coordinate spaziali

$$y(\theta, x) = a_3 x^2 + b_3 x + c_3 \quad (3)$$

per ciascun angolo con l'obiettivo di individuare le curve di regressione che meglio approssimano le traiettorie osservate sperimentalmente. Questa procedura consente di confrontare le forme delle traiettorie empiriche con quelle previste dai modelli teorici facilitando l'identificazione di eventuali deviazioni dovute a fattori sperimentali come attrito o errori sistematici. I risultati numerici e i parametri ottenuti dalle elaborazioni sono riportati in dettaglio nella tabella 1 che fornisce una base quantitativa solida per la validazione dei principi fisici e matematici alla base del moto parabolico.

Raggio circonferenza punti di lancio $r_0 = 0.11454\text{ m}$			
θ	$x(\theta, t) = b_1 t + c_1, y(\theta, t) = a_2 t^2 + b_2 t + c_2,$ $y(\theta, x) = a_3 x^2 + b_3 x + c_3$		
Intervallo di fit (s) Punto di lancio (m)	a_2, a_3	b_1, b_2, b_3	c_1, c_2, c_3
35°			
$0.1 \leq t \leq 0.55$	-4.921 ± 0.002	3.5665 ± 0.0003	-0.1053 ± 0.0001
$(0.09542, 0.06354)$	-0.3866 ± 0.0002	3.027 ± 0.002	-0.0947 ± 0.0002
		0.7667 ± 0.0004	-0.0093 ± 0.0002
45°			
$0.1 \leq t \leq 0.65$	-4.881 ± 0.002	2.9820 ± 0.0007	-0.0838 ± 0.0003
$(0.08051, 0.08159)$	-0.5484 ± 0.0004	3.669 ± 0.001	-0.1135 ± 0.0002
		1.1365 ± 0.0008	-0.0129 ± 0.0004
55°			
$0.1 \leq t \leq 0.75$	-4.905 ± 0.003	2.3908 ± 0.0003	-0.0821 ± 0.0001
$(0.06330, 0.09546)$	-0.8570 ± 0.0004	4.184 ± 0.003	-0.1510 ± 0.0005
		1.6073 ± 0.0008	-0.0128 ± 0.0003
65°			
$0.1 \leq t \leq 0.8$	-4.896 ± 0.003	1.7252 ± 0.0002	-0.0433 ± 0.0001
$(0.04667, 0.10463)$	-1.642 ± 0.001	4.456 ± 0.003	-0.1327 ± 0.0006
		2.432 ± 0.002	-0.0204 ± 0.0007
75°			
$0.1 \leq t \leq 0.9$	-4.912 ± 0.004	1.0274 ± 0.0005	-1.0274 ± 0.0005
$(0.02718, 0.11151)$	-4.644 ± 0.007	4.730 ± 0.004	-0.1559 ± 0.0009
		4.358 ± 0.007	-0.040 ± 0.002
85°			
$0.1 \leq t \leq 0.9$	-4.875 ± 0.003	0.282 ± 0.001	-0.0004 ± 0.0006
$(0.00596, 0.11438)$	-60.7 ± 0.5	4.841 ± 0.003	-0.1683 ± 0.0006
		16.8 ± 0.1	-0.131 ± 0.008

Tab. 1 Dati ricavati col Tracker analizzando le tracce delle traiettorie $(x(\theta, t), y(\theta, t), y(\theta, x))$ e stimando i punti di lancio per i valori impostati di θ .

I dati sperimentali raccolti mostrano un'eccellente concordanza con le equazioni teoriche delle parabole validando in modo rigoroso la natura parabolica del moto dei proiettili. Questo risultato conferma sperimentalmente quanto già descritto nei modelli classici del moto balistico, risalenti agli studi di Galileo Galilei, rafforzando l'affidabilità delle leggi fisiche che governano la traiettoria di un corpo soggetto esclusivamente all'azione della gravità in assenza di resistenza dell'aria. L'intero processo di analisi è stato condotto con particolare attenzione e rigore metodologico al fine di assicurare la massima affidabilità dei risultati ottenuti e delle conclusioni elaborate a partire dalle traiettorie registrate.

2.3 Valutazione della curva di sicurezza

Nella fase conclusiva dell'esperienza le traiettorie della biglia in forma analitica sono state analizzate utilizzando il software *Geogebra* [12], uno strumento che consente di elaborare e visualizzare in modo intuitivo concetti geometrici.

L'obiettivo di questa sezione è chiarire le assunzioni di base impiegate nell'analisi, esplicitare il passaggio dalle equazioni teoriche ai dati sperimentali e comprendere le principali fonti di errore.

- Le traiettorie analizzate rappresentano il moto di un proiettile lanciato da un punto con una velocità iniziale $\mathbf{v}_0$ e soggetto all'accelerazione costante $\mathbf{g}$ dovuta alla gravità terrestre. L'assunzione di un'accelerazione costante è fondamentale, poiché consente di applicare le leggi del moto parabolico e di descrivere il fenomeno in modo semplice e rigoroso. Tale ipotesi garantisce che la traiettoria seguita dalla biglia sia una parabola, in pieno accordo con i principi della fisica classica. È importante sottolineare che la precisione del lanciatore riveste un ruolo cruciale: se quest'ultimo non fosse sufficientemente accurato nel garantire condizioni iniziali costanti, la validità di questa approssimazione verrebbe meno, compromettendo la corrispondenza tra i modelli teorici e i dati sperimentali raccolti.
- Il moto avviene all'interno del piano definito dai vettori $\mathbf{v}_0$ e $\mathbf{g}$, corrispondente al piano (x, y) inquadrato nelle riprese, nel quale l'asse y risulta orientato in direzione opposta rispetto al vettore accelerazione $\mathbf{g}$. Questa scelta permette di adottare un sistema di riferimento unificato per la rappresentazione di tutte le traiettorie, rendendo più semplice e immediato il confronto tra i diversi lanci effettuati. Tuttavia, l'affidabilità di tale approccio è strettamente legata alla precisione delle

registrazioni video e all'accurata determinazione delle coordinate spaziali. Eventuali imprecisioni nell'orientamento del piano possono infatti provocare distorsioni nella misura degli angoli e delle distanze, influenzando negativamente la qualità dei dati raccolti e la validità delle conclusioni sperimentali.

- Si assume che il modulo della velocità iniziale v_0 sia costante per tutti i lanci, ovvero la molla fornisce alla biglia sempre la stessa quantità di energia e impulso, ciò che varia è esclusivamente la direzione determinata dall'angolo di lancio θ . Questa ipotesi rappresenta un presupposto fondamentale per poter confrontare in modo coerente e significativo le diverse traiettorie. Se infatti la velocità iniziale non fosse identica in ogni prova le differenze osservate nei percorsi seguiti dalla biglia potrebbero essere attribuite non solo all'angolo di lancio ma anche a variazioni dell'energia trasmessa. Di conseguenza diverrebbe molto più complesso distinguere l'effetto dell'angolo da quello della velocità e validare correttamente i modelli teorici alla base del moto parabolico.
- Affinché l'analisi sia coerente con le condizioni ideali descritte nel caso affrontato da Torricelli è fondamentale che tutte le traiettorie abbiano un'origine comune. Questo comporta una traslazione geometrica delle traiettorie sperimentali, operazione che non modifica in alcun modo la cinematica del fenomeno ma permette di uniformare il punto di partenza dei diversi lanci. In questo modo si crea la possibilità di sovrapporre e confrontare direttamente le varie curve isolando con precisione l'effetto dell'angolo di lancio. Tale procedimento facilita l'analisi delle cosiddette parabole di sicurezza e rende più chiara la valutazione delle differenze tra i dati sperimentali e le previsioni teoriche.

Alla luce delle ipotesi sopra descritte le equazioni che regolano la posizione della biglia in funzione del tempo assumono la seguente forma

$$\begin{cases} x = v_0 \cos(\theta) t \\ y = v_0 \sin(\theta) t - \frac{1}{2} g t^2 \end{cases} \quad (4)$$

con v_0 , g lunghezze dei corrispondenti vettori $\mathbf{v}_0$, $\mathbf{g}$. In forma esplicita:

$$y = \tan(\theta) x - \frac{1}{2} \frac{g}{v_0^2 \cos^2(\theta)} x^2 \quad (5)$$

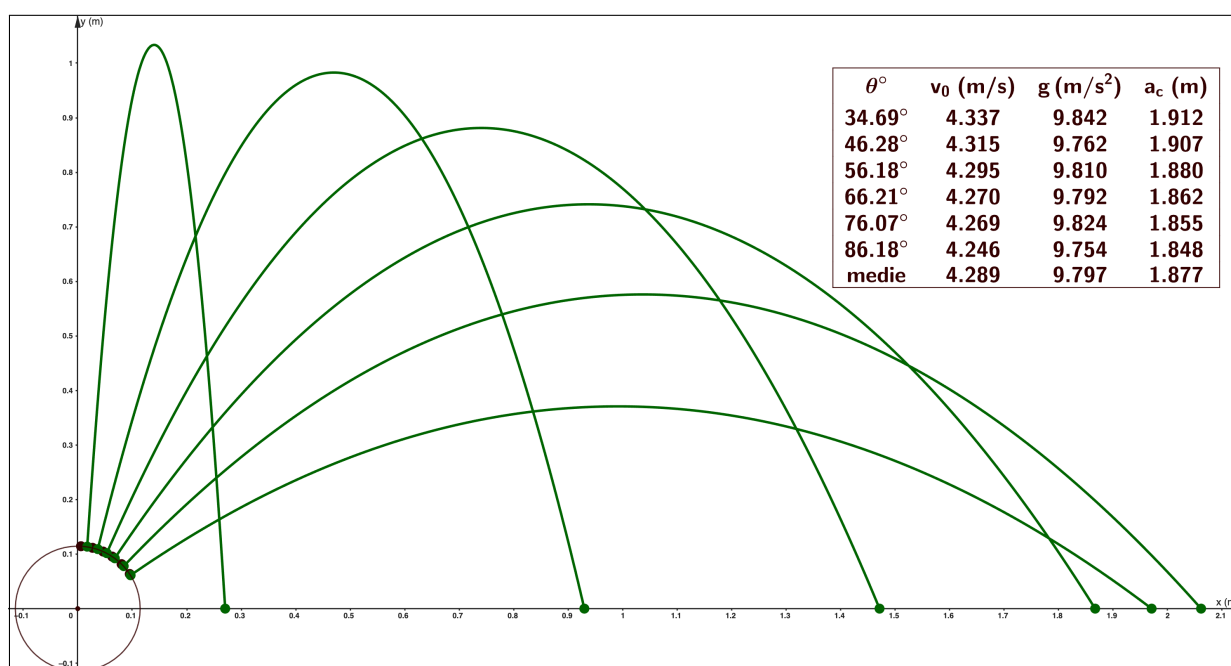


Fig. 6 Tracce delle traiettorie.

con θ inclinazione tra $\mathbf{v}_0$ e l'asse x . Come scoperto da Torricelli con metodi puramente geometrici (per una trattazione basata sulla geometria analitica si veda la sezione 3.1) la curva di sicurezza risulta

$$\begin{aligned} y &= \frac{1}{2} \frac{v_0^2}{g} - \frac{1}{2} \frac{g}{v_0^2} x^2 \\ &= \frac{a_c}{2} - \frac{1}{2 a_c} x^2, \quad a_c = \frac{v_0^2}{g} \end{aligned} \quad (6)$$

dove abbiamo inserito il parametro caratteristico a_c che rappresenta fisicamente la gittata del cannone cioè la distanza orizzontale massima raggiunta dal proiettile alla stessa quota del lancio.

La figura 6 mostra le traiettorie ottenute tramite il software *Tracker*, tali curve, secondo i dati di fitting, corrispondono ad archi parabolici. Sono inoltre indicati (in nero) i presunti punti di lancio individuati nei fotogrammi mediante appositi contrassegni sul lanciatore, che giacciono su una circonferenza centrata nel perno del cannoncino.

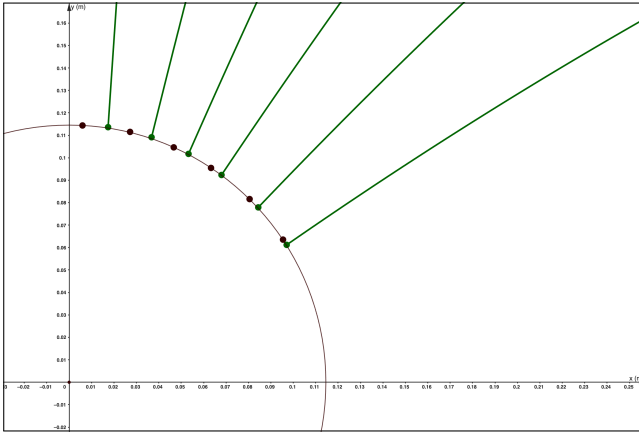


Fig. 7 Ingrandimento delle origini delle traiettorie. Analizzando la zona circostante l'origine (figura 7) si osserva che le traiettorie non passano esattamente per i punti individuati dal lanciatore (punti neri). Questo può essere attribuito al fatto che l'impatto del martelletto non risulta perfettamente allineato con il baricentro della biglia, introducendo così errori sistematici; inoltre, eventuali rimbalzi contro la parete interna del cannoncino possono generare errori casuali aggiuntivi.

Grazie alle equazioni orarie e all'utilizzo di *Geogebra* è stato possibile individuare con precisione i punti delle traiettorie più vicini a quelli corrispondenti individuati sul lanciatore. Questi punti sono stati assunti come iniziali per le traiettorie (evidenziati nelle figure 6 e 7) e hanno permesso di stimare anche il momento esatto del lancio. Dall'analisi dei dati emerge che la discrepanza tra i punti teorici e quelli effettivamente osservati tende ad aumentare leggermente all'aumentare dell'angolo di lancio, segnalando una maggiore sensibilità agli errori sistematici per traiettorie più inclinate.

Per verificare la corrispondenza tra modelli teorici e osservazioni pratiche il software *Geogebra* ha permesso la valutazione dei parametri caratteristici delle traiettorie:

- Angolo di lancio (θ) effettivo: determinato analizzando l'inclinazione iniziale di ciascuna traiettoria rispetto all'asse x . L'angolo influenza direttamente la forma della parabola e la gittata raggiunta dal proiettile.
- Velocità iniziale (v_0): una velocità iniziale abbastanza uniforme conferma la validità delle ipotesi e permette di attribuire le variazioni osservate esclusivamente all'angolo di lancio.
- L'accelerazione di gravità (g): i valori concordano abbastanza bene con il valore medio 9.810 m/s^2 considerato valido sulla superficie terrestre e con la stima 9.785 m/s^2 alla latitudine locale.
- Parametro caratteristico (a_c): rappresenta la gittata massima raggiungibile dal proiettile, ossia la distanza orizzontale raggiunta alla stessa quota di partenza.

θ	$v_0 \text{ (m/s)}$	$g \text{ (m/s}^2\text{)}$	$a_c \text{ (m)}$
34.69°	4.337	9.842	1.912
46.28°	4.315	9.762	1.907
56.18°	4.295	9.810	1.880
66.21°	4.270	9.792	1.862
76.07°	4.269	9.824	1.855
86.18°	4.246	9.754	1.848
medie	4.289	9.797	1.877

Tab. 2 Caratteristiche cinematiche delle traiettorie.

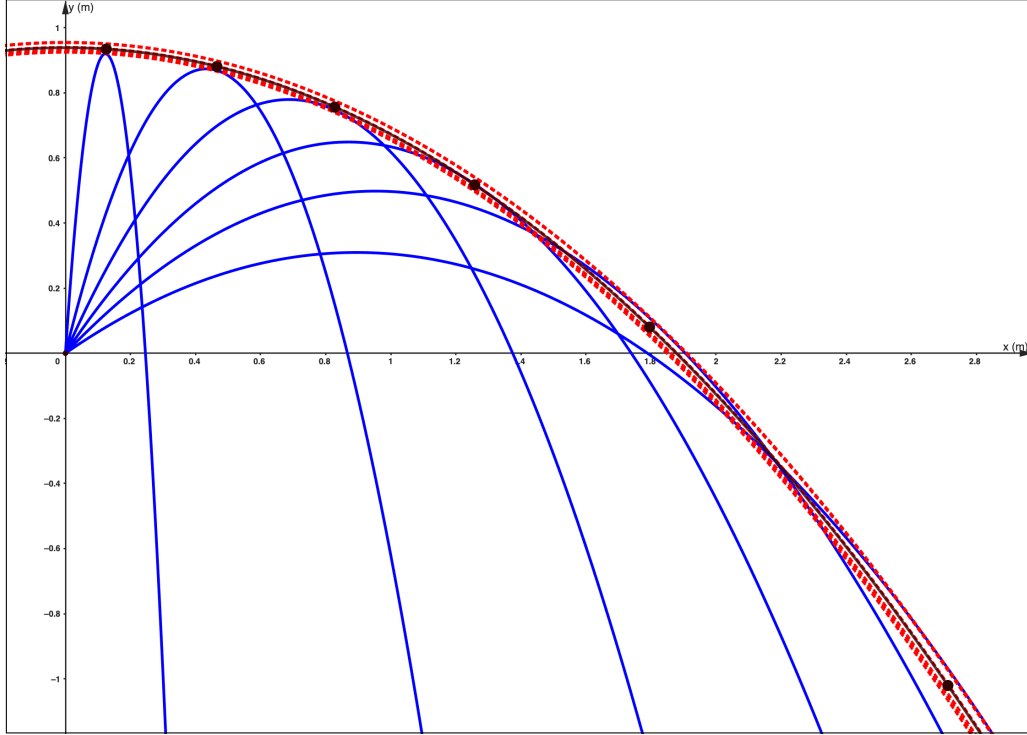


Fig. 8 Tracce delle traiettorie con corrispondenti parabole di sicurezza e la curva di sicurezza media.

I valori sono mostrati nell'inserito della figura 6 e nella tabella 2 con l'ultima riga che rappresenta la media dei valori sovrastanti.

Ogni traiettoria è stata traslata spazialmente e temporalmente assumendo il punto e momento di lancio come origine delle coordinate e del tempo. In questo modo ci siamo ricondotti a un punto comune di partenza come il caso studiato da Torricelli.

Nella figura 8 si vedono le traiettorie (in blu) con l'origine in comune. Per ogni traiettoria possiamo calcolare quale sarebbe la sua parabola di sicurezza (queste sono tratteggiate in rosso). La parabola di sicurezza media (con $a_c = 1.877$, media dei parametri caratteristici) è rappresentata in nero. Le parabole tratteggiate in rosso evidenziano visivamente l'incertezza sperimentale. I punti evidenziati in nero rappresentano i punti di contatto teorici con le singole traiettorie considerate.

Si può considerare anche il caso delle traiettorie originali sincronizzate solo temporalmente senza essere traslate nell'origine. Non cambiano i parametri cinematici, ma solo i punti di partenza che sono situati su un arco di circonferenza di raggio $r_0 = 0.1154 m$. L'analisi precedente può essere considerata valida se la lunghezza r_0 è trascurabile rispetto alle lunghezze in gioco (come si era ritenuto erroneamente all'inizio) ma occorre una valutazione diversa della curva di sicurezza. In questo caso le equazioni orarie del moto risultano

$$\begin{cases} x = (r_0 + v_0 t) \cos(\theta) \\ y = (r_0 + v_0 t) \sin(\theta) - \frac{1}{2} g t^2 \end{cases} \quad (7)$$

ma la curva di sicurezza non è più una parabola (anche se molto simile) e non è possibile determinare una forma esplicita univoca ma solo in forma parametrica

$$\begin{cases} x = r_0 \cos(\theta) + a_c \frac{1}{\tan(\theta)} \\ y = r_0 \sin(\theta) + \frac{1}{2} a_c \left(1 - \frac{1}{\tan^2(\theta)} \right) \end{cases} \quad (8)$$

Si veda la sezione 3.2 per i dettagli.

La figura 9 mostra le parabole in verde con le corrispondenti curve di sicurezza tratteggiate in rosso e la curva di sicurezza media (in nero) evidenziando su di essa i punti di contatto teorici con le traiettorie.

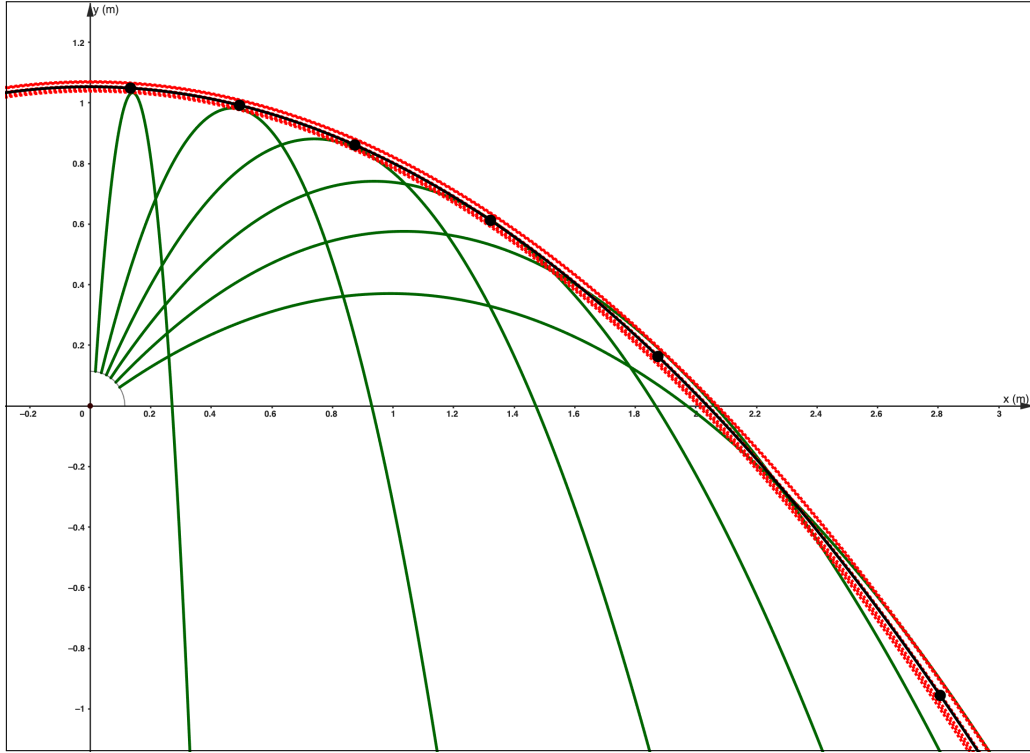


Fig. 9 Traiettorie originali non traslate e curve di sicurezza.

L'accordo delle varie curve involuppo risulta molto buono, leggermente meglio del caso precedente, considerando che ci sono sicuramente errori dovuti a cause non valutabili o non rimediabili con i mezzi a disposizione.

3 Appendice matematica: curve e involuppi

Le parabole studiate da Torricelli costituiscono matematicamente una *famiglia di curve* dipendenti da un unico parametro (l'angolo di lancio θ) e la parabola di sicurezza è la *curva involuppo* di tale famiglia. Torricelli ha dedotto nella sua opera uno dei primi esempi non banali di curva involuppo utilizzando solo la geometria e in questa sezione si descrivono brevemente il concetto e le metodologie che sono state sviluppate successivamente a Torricelli.

Un insieme di curve piane

$$\Phi = \left\{ \varphi_{\theta} \text{ curva regolare in } \mathbb{R}^2, \theta \in [\theta_1, \theta_2] \right\} \quad (9)$$

è detto una *famiglia di curve* regolari parametrizzate da un parametro θ reale, e una curva regolare γ è detta un *involuppo* per la famiglia Φ , se per ogni punto P di γ passa una ed una sola curva $\varphi_{\theta} \in \Phi$, la quale risulti anche tangente in P a γ .

Una curva è regolare se in ogni suo punto è ben definita in maniera univoca una retta tangente. Per semplicità si assume che tutte le funzioni considerate in seguito siano, almeno a tratti, continue e derivabili con continuità.

3.1 Involuppo tramite equazioni cartesiane

Se la famiglia Φ è espressa tramite equazioni cartesiane come luogo di punti nel piano

$$\varphi_{\theta} : \left\{ (x, y) \in \mathbb{R}^2 ; f(x, y, \theta) = 0 \right\}, \quad \theta_1 \leq \theta \leq \theta_2, \quad (10)$$

con f che definisce una curva regolare nel piano (x, y) a θ fissato, possiamo ragionare come segue per ricercare la curva involuppo γ della famiglia Φ .

Per ogni valore di θ abbiamo un punto della curva γ (che assumiamo unico) $P(\theta)$ di contatto con la curva φ_θ in cui la tangente in $P(\theta)$ a γ coincide con la tangente a φ_θ . Possiamo allora pensare γ parametrizzata dal parametro θ tramite il punto di contatto

$$\gamma : \theta \longrightarrow P(\theta) = (x(\theta), y(\theta)), \quad (11)$$

con $x(\theta)$ e $y(\theta)$ funzioni, per ora incognite, che definiscono le equazioni parametriche di γ

$$\gamma : \begin{cases} x = x(\theta) \\ y = y(\theta) \end{cases} \quad \theta_1 \leq \theta \leq \theta_2. \quad (12)$$

Inoltre poichè $P(\theta)$ appartiene alla curva φ_θ si deve avere identicamente per ogni θ

$$f(x(\theta), y(\theta), \theta) = 0. \quad (13)$$

La tangente nel punto $P(\theta)$ a φ_θ ha equazione

$$\frac{\partial f}{\partial x}(x(\theta), y(\theta), \theta) (x - x(\theta)) + \frac{\partial f}{\partial y}(x(\theta), y(\theta), \theta) (y - y(\theta)) = 0 \quad (14)$$

mentre la tangente a γ ha equazione

$$\frac{dy}{d\theta}(\theta) (x - x(\theta)) = \frac{dx}{d\theta}(\theta) (y - y(\theta)). \quad (15)$$

Le due equazioni rappresentano la stessa retta se e solo se (omettiamo per semplicità gli argomenti ovvi)

$$\frac{\partial f}{\partial x} \frac{dx}{d\theta} + \frac{\partial f}{\partial y} \frac{dy}{d\theta} = 0. \quad (16)$$

Derivando totalmente la relazione (13) si deve avere identicamente per ogni θ :

$$\frac{df}{d\theta} = \frac{\partial f}{\partial x} \frac{dx}{d\theta} + \frac{\partial f}{\partial y} \frac{dy}{d\theta} + \frac{\partial f}{\partial \theta} = 0 \quad (17)$$

pertanto (senza più coinvolgere le derivate di x e y rispetto a θ)

$$\frac{\partial f}{\partial \theta} = 0. \quad (18)$$

In conclusione la regola pratica è di risolvere il sistema di equazioni

$$\begin{cases} f(x, y, \theta) = 0 \\ \frac{\partial f}{\partial \theta}(x, y, \theta) = 0. \end{cases} \quad (19)$$

Possiamo eliminare il parametro θ risolvendo una delle due equazioni, $\theta = \theta(x, y)$ e sostituirlo nell'altra ottenendo l'equazione dell'involuppo γ in forma cartesiana, oppure risolvere il sistema ricavando x e y in funzione di θ , ricavando la curva γ in forma parametrica.

Considerando come esempio la famiglia di parabole (5) studiate da Torricelli abbiniamo tale equazione alla sua derivata parziale rispetto a θ per ottenere il sistema che fornisce l'involuppo

$$\begin{cases} y = \tan(\theta) x - \frac{1}{2} \frac{g}{v_0^2 \cos^2(\theta)} x^2 \\ 0 = \frac{x}{\cos^2(\theta)} - \frac{g}{v_0^2} \frac{\tan(\theta)}{\cos^2(\theta)} x^2. \end{cases} \quad (20)$$

Eliminando il fattore $x/\cos^2(\theta)$ (che supponiamo non nullo) abbiamo dalla seconda equazione

$$\tan(\theta) = \frac{v_0^2}{g} \frac{1}{x} \quad (21)$$

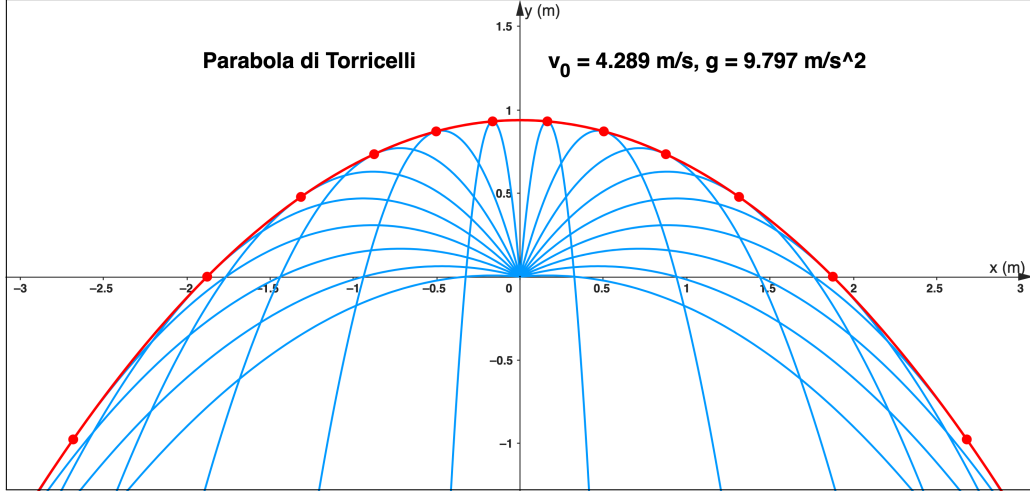


Fig. 10 Parabole balistiche e la parabola di sicurezza di Torricelli. I punti indicano i contatti tra l'involuppo e le parabole.

da cui la parabola di sicurezza

$$y = \frac{1}{2} \frac{v_0^2}{g} - \frac{1}{2} \frac{g}{v_0^2} x^2 \quad (22)$$

oppure in forma parametrica

$$\begin{cases} x = \frac{v_0^2}{g} \tan(\theta) \\ y = \frac{1}{2} \frac{v_0^2}{g} \left(1 - \frac{1}{\tan^2(\theta)} \right) \end{cases} \quad (23)$$

Nella figura 10 vediamo un esempio di traiettorie paraboliche con la relativa parabola di sicurezza. I valori dei parametri sono scelti in base alla sezione 2.3.

3.2 Inviluppo tramite equazioni parametriche

Se la famiglia Φ è espressa tramite equazioni parametriche:

$$\varphi_\theta : t \longrightarrow (x(\theta, t), y(\theta, t)), \quad t_1 \leq t \leq t_2, \quad \theta_1 \leq \theta \leq \theta_2. \quad (24)$$

con $x(\theta, t)$, $y(\theta, t)$ funzioni continue e derivabili. La regolarità impone che le derivate $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$ esistano e non siano entrambe nulle.

Se esiste una curva inviluppo γ della famiglia Φ per ogni punto P della curva inviluppo abbiamo un solo valore di θ che individua la curva $\varphi_\theta \in \Phi$ tangente per un valore (che assumiamo unico) del parametro $t = t(\theta)$. Possiamo quindi parametrizzare la curva γ tramite il parametro θ stesso

$$\gamma : \theta \longrightarrow P(\theta) = (x(\theta, t(\theta)), y(\theta, t(\theta))), \quad \theta_1 \leq \theta \leq \theta_2 \quad (25)$$

e assumiamo che la curva γ sia anch'essa regolare. Affinchè la curva γ risulti l'inviluppo della famiglia Φ le tangenti in P alla curva φ_θ e alla curva γ devono coincidere.

La tangente alla curva φ_θ è individuata dalla sua dipendenza da t (con θ fissato), con direzione del vettore

$$\mathbf{v} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) \Big|_{(\theta, t(\theta))} \quad (26)$$

mentre la tangente alla curva γ è individuata dalla sua dipendenza totale da θ

$$\mathbf{w} = \left(\frac{dx}{d\theta}, \frac{dy}{d\theta} \right) \Big|_{(\theta, t(\theta))} = \left(\frac{\partial x}{\partial \theta} + \frac{\partial x}{\partial t} \frac{dt}{d\theta}, \frac{\partial y}{\partial \theta} + \frac{\partial y}{\partial t} \frac{dt}{d\theta} \right) \Big|_{(\theta, t(\theta))}. \quad (27)$$

Affinchè le due tangenti coincidano i due vettori $\mathbf{v}$ e $\mathbf{w}$ (che assumiamo non nulli) devono essere proporzionali tra loro (il determinante delle loro componenti è nullo)

$$\frac{\partial x}{\partial t} \left(\frac{\partial y}{\partial \theta} + \frac{\partial y}{\partial t} \frac{dt}{d\theta} \right) = \frac{\partial y}{\partial t} \left(\frac{\partial x}{\partial \theta} + \frac{\partial x}{\partial t} \frac{dt}{d\theta} \right), \quad (28)$$

$$\frac{\partial x}{\partial t} \frac{\partial y}{\partial \theta} = \frac{\partial y}{\partial t} \frac{\partial x}{\partial \theta}$$

(sono stati omessi gli argomenti delle funzioni per non appesantire le notazioni). Pertanto la curva γ è individuata dal sistema di equazioni

$$\begin{cases} x = x(\theta, t) \\ y = y(\theta, t) \\ \frac{\partial x(\theta, t)}{\partial \theta} \frac{\partial y(\theta, t)}{\partial t} - \frac{\partial y(\theta, t)}{\partial \theta} \frac{\partial x(\theta, t)}{\partial t} = 0. \end{cases} \quad (29)$$

In pratica l'ultima equazione permette di eliminare uno dei due parametri, t in funzione di θ o viceversa, se è soddisfatto il requisito di unicità dei punti di contatto e regolarità delle curve, e si ottengono delle equazioni parametriche in forma canonica.

Come esempio trattiamo il caso dell'equazione (7) in cui l'apparato di lancio sia un cannone la cui canna possa ruotare attorno a un perno regolando l'angolo θ di uscita del proiettile ma questo perno non coincide con l'estremità di uscita stessa.

Abbiamo una famiglia di parabole leggermente diversa dall'esempio precedente (che ritroviamo per $r_0 = 0$) e per determinare l'inviluppo dobbiamo aggiungere alle equazioni (7) la condizione:

$$\frac{\partial x}{\partial t} \frac{\partial y}{\partial \theta} = \frac{\partial y}{\partial t} \frac{\partial x}{\partial \theta}. \quad (30)$$

Quindi:

$$(r_0 + v_0 t) [v_0 \cos^2(\theta) + v_0 \sin^2(\theta) - g t \sin(\theta)] = 0. \quad (31)$$

Assumendo $r_0 + v_0 t \neq 0$ otteniamo:

$$v_0 - g t \sin(\theta) = 0, \quad (32)$$

da cui possiamo ricavare t :

$$t = \frac{v_0}{g \sin(\theta)}, \quad |t| \geq \frac{v_0}{g} \quad (33)$$

ottenendo le equazioni parametriche della curva inviluppo:

$$\begin{cases} x = r_0 \cos(\theta) + \frac{v_0^2}{g} \frac{1}{\tan(\theta)} \\ y = r_0 \sin(\theta) + \frac{1}{2} \frac{v_0^2}{g} \left(1 - \frac{1}{\tan^2(\theta)} \right) \end{cases} \quad (34)$$

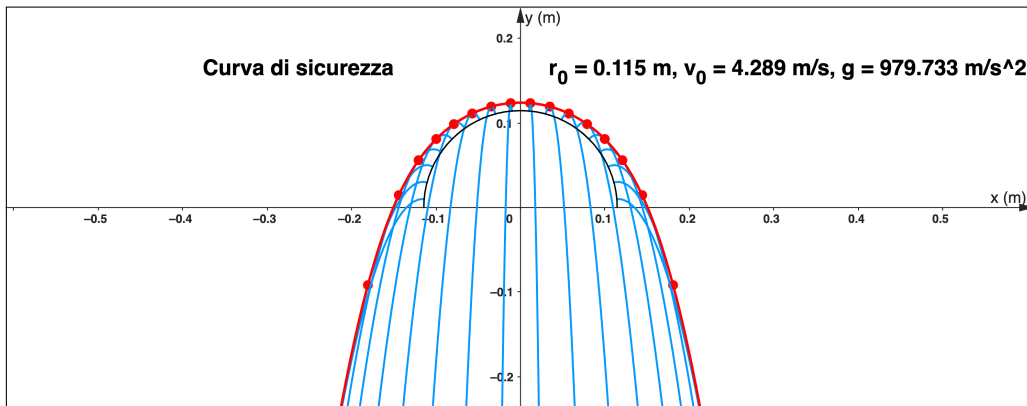


Fig. 11 Parabole balistiche e curva di sicurezza con una gravità molto elevata. I punti indicano i contatti tra l'inviluppo e le parabole.

La curva di sicurezza non è più una parabola; vedi in particolare la figura 11 con i parametri sono scelti in base alla sezione 2.3 ma con un valore molto alto dell'accelerazione di gravità. Un tratto della curva di sicurezza coincide praticamente con la circonferenza descritta dalla canna del cannone e prosegue in direzione verticale. Inoltre non è semplice eliminare il parametro θ dalle equazioni parametriche.

4 Conclusione

La parabola di sicurezza di Torricelli rappresenta un pilastro della fisica classica e della balistica, incastonandosi in un contesto storico di grandi scoperte scientifiche. Comprendere i suoi principi permette di cogliere non solo i confini imposti dalle leggi naturali ai movimenti dei corpi, ma anche la profondità e la lungimiranza di uno dei più importanti scienziati del Seicento.

La parabola di sicurezza di Torricelli non è soltanto una formula da conoscere, ma un vero e proprio simbolo della genialità e della capacità umana di interpretare il mondo. Questa curva, infatti, segna un passaggio fondamentale nella storia della scienza: rappresenta uno dei primi tentativi di applicare i principi matematici allo studio dei fenomeni fisici reali, sancendo l'incontro tra teoria e pratica. La sua importanza scientifica risiede nel fatto che ha permesso di comprendere e prevedere con precisione i limiti imposti dalla natura al movimento dei corpi, gettando le basi della balistica moderna e rivoluzionando la progettazione di fortificazioni e strategie militari.

Nel corso dell'esperimento sono emersi sia errori sistematici sia errori casuali che hanno influenzato la precisione delle misurazioni e la qualità dell'accordo tra teoria e pratica.

- Errori sistematici: derivano da fattori strutturali dell'apparato sperimentale, come l'allineamento imperfetto tra il martelletto della molla e il baricentro della biglia, oppure la presenza di una circonferenza di punti di lancio anziché un'origine esatta comune. Questi errori si ripetono in modo simile in ogni lancio e possono essere ridotti migliorando la calibrazione dell'apparato, curando l'allineamento meccanico e verificando regolarmente la posizione del lanciatore.
- Errori casuali: sono dovuti a fattori imprevedibili e variabili, come piccoli rimbalzi interni della biglia, fluttuazioni nella forza impressa dalla molla o imprecisioni nella rilevazione dei punti di lancio nei fotogrammi. Questi errori possono essere stimati calcolando la deviazione standard dei parametri misurati che fornisce una misura quantitativa dell'incertezza associata alle osservazioni. Per ridurre l'impatto degli errori si suggerisce:
- Effettuare più lanci per ciascun angolo e calcolare i valori medi e la deviazione standard dei parametri.
- Utilizzare strumenti di misura e software di analisi ben calibrati e verificare la qualità delle riprese video.
- Applicare procedure di fitting robuste che tengano conto delle possibili variazioni nei dati.

L'analisi delle traiettorie tramite *Geogebra*, basata su assunzioni esplicite e motivate, ha permesso di verificare sperimentalmente la validità delle equazioni del moto parabolico e della parabola di sicurezza di Torricelli. Il confronto tra modelli teorici e dati sperimentali ha mostrato una buona corrispondenza nonostante la presenza di errori sistematici e casuali. L'approfondimento delle fonti di errore e l'applicazione di strategie di riduzione e stima dell'incertezza hanno rafforzato l'affidabilità dei risultati ottenuti. Nel complesso l'esperienza ha evidenziato l'importanza di un approccio rigoroso e critico nell'analisi dei dati, fornendo agli studenti una metodologia efficace per la validazione sperimentale dei principi fisici.

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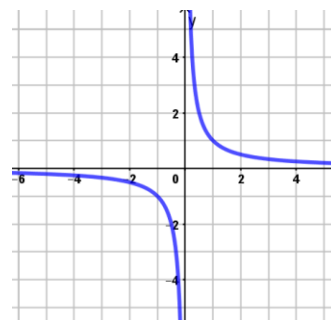
Il ‘solido iperbolico acuto’ nell’Opera Geometrica di Torricelli: alcune riflessioni tra presente e passato

Lorenza Resta
Società Torricelliana di Scienze e Lettere
Liceo Torricelli, Faenza

Questo lavoro è un estratto dall’articolo pubblicato nel “Il Nuovo Nautilus 2025”, frutto dello studio svolto in occasione del Convegno “Torricelli ieri e oggi”.

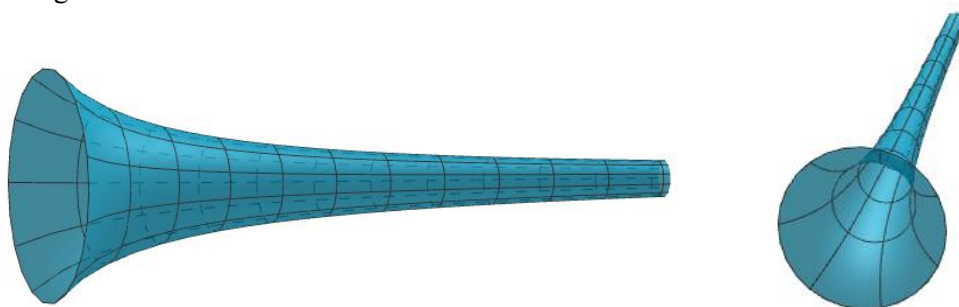
Definizione del solido acuto iperbolico e sua rappresentazione

Molto famosa è la legge di proporzionalità inversa tra due grandezze x e y , la proprietà che caratterizza tale legge è il prodotto sempre costante tra le due grandezze: $x \cdot y = \text{costante}$ (un esempio di grandezze legate da proporzionalità inversa è la misura di base e altezza per rettangoli che abbiano la stessa area). La curva che ha come equazione la legge della proporzionalità inversa viene chiamata iperbole equilatera riferita agli asintoti e il suo grafico è riportato in figura (nel caso di costante positiva).



Se si considera un ramo di iperbole equilatera riferita agli asintoti e lo si mette in rotazione attorno ad uno dei suoi asintoti si ottiene la superficie di un solido che Evangelista Torricelli chiamò “solido iperbolico acuto” nella sua Opera Geometrica pubblicata nel 1644.

Nel dettaglio si può considerare il ramo definito dall’equazione $y = \frac{1}{x}$ con la restrizione $x \geq 1$ e lo si mette in rotazione attorno all’asse delle ascisse (uno dei suoi asintoti) e si ottiene il grafico riportato nella figura seguente.



E’ possibile trovare immagini molto simili al “solido iperbolico acuto” anche chiamato “Tromba di Torricelli o di Gabriele” in molti dipinti, come per esempio “Il Giudizio Universale” riportato sotto.



Estratto da “Il Giudizio Universale” di Felice Damiani, 1575-1609 Gubbio

Oggigiorno sono disponibili software didattici che permettono di realizzare immagini di molti solidi; in particolare il software *Geogebra* permette sia di visualizzare i solidi sia di generarli come superfici di rotazione, ottenute ruotando una curva attorno ad una retta voluta.

Alcuni elementi storici relativi al solido acuto iperbolico e ai suoi ideatori.

Questo solido ha affascinato molti matematici nell'arco della storia, Evangelista Torricelli ne ha descritto alcuni aspetti importanti nella sua Opera Geometrica, suscitando stima e ammirazione da parte di altri matematici del tempo. Uno fra tutti il matematico Cavalieri, in una missiva scritta a Torricelli, parla di «saporitissimi frutti del suo ingegno, essendomi riuscito infinitamente ammirabile quel solido iperbolico infinitamente lungo ed eguale ad un corpo quanto a tutte e tre le dimensioni finito[...].» Ancora Cavalieri nella stessa missiva scrive «[...] la ringrazio della dimostrazione del solido acuto iperbolico veramente divina [...]».

Nel testo di D'Amore e Sbaragli, "La matematica e la sua storia" si può leggere «*Fertile e attento Torricelli fu senza dubbio il più brillante allievo di Galilei. [...] Torricelli accettò il potente mezzo algoritmico (metodo degli indivisibili) ideato dal frate gesuato e ne diventò uno dei più convinti applicatori e sviluppatori*». Ancora, nello stesso testo: «*Riferendosi al suo principio degli indivisibili curvi, (Torricelli) immagina il solido costituito dagli infinitesimi rappresentati dalle infinite superfici cilindriche di altezza x e raggio $1/x$* ».

Non è stato però Torricelli il primo matematico ad essersi occupato di questo solido e delle sue proprietà; già il matematico francese Nicola D'Oresme ne aveva colto le peculiarità ma i suoi risultati non erano noti a Torricelli e ai matematici del tempo, della metà del Seicento.

Già dalle parole della lettera di Cavalieri a Torricelli emergono alcune delle proprietà sbalorditive di questo solido: possiede una superficie infinita ma racchiude una regione di spazio di volume finito. In alcune fonti reperibili in rete e anche nel testo "A piccole dosi: contro la crisi di astinenza dalla matematica" del matematico Pergiorio Odifreddi, le caratteristiche del solido acuto iperbolico vengono descritte in maniera approssimativa con la dicitura «paradosso del pittore: per verniciare la superficie esterna del solido servirebbe una quantità infinita di vernice mentre lo stesso solido conterrebbe al suo interno solo una quantità finita di vernice ».

La parola "paradosso" deriva dal greco antico "paradoxos", che è composta da "para-", che significa "contro", e "doxa", che significa "opinione"; infatti, un paradosso è una proposizione che sembra contraria al senso comune ma che se esaminata con maggiore profondità, in molti casi, porta a risultati validi.

I paradossi hanno avuto un ruolo importante nel corso della storia e della filosofia e alcuni di questi sono collegati al tema dell'infinito, come nel caso del paradosso del pittore descritto sopra. Alcuni paradossi relativi all'infinito sono antichi come, per esempio, il famoso paradosso di Zenone di Elea di Achille e la Tartaruga; altri invece sono moderni come il celebre paradosso del Grand Hotel di Hilbert. Molti paradossi hanno portato a sviluppi e conseguenze importanti nella storia matematica.

Ancora nel testo "La matematica e la sua storia" si può leggere «Torricelli, dunque, sebbene non formalmente aveva già chiari quei concetti che avrebbero portato alle idee di limite e di infinitesimo. Bonaventura Cavalieri, Pierre De Fermat, Gilles Personne de Roberval e Cristiaan Hygens e tanti altri ancora applaudirono al suo genio. Roberval afferma perfino che l'opera di Torricelli era la più ardita e la più feconda, quella che aveva aperto la strada ai successivi progressi [...] Possiamo affermare che Torricelli tratta esplicitamente in senso attuale l'infinito [...] Nella sua creazione del calcolo integrale Gottfried Wilhelm von Leibniz citerà come suo precursore proprio Torricelli... Ancora di Torricelli sono la prima rettificazione di una curva generica e il calcolo del primo integrale definito su un campo illimitato. Quello che in lui manca è, a nostro avviso, semplicemente una presa di coscienza del fatto nuovo, del calcolo potente, rapido e rigoroso che stava nascendo».

Mentre nel testo di Giorgio Tomaso Bagni, "Storia della Matematica" si può leggere «Torricelli, dunque, intuì alcune tra le più importanti caratteristiche teoriche del calcolo infinitesimale; ma la sua opera non ebbe la diffusione che avrebbe consentito ad essa di incidere in termini decisivi nella storia dell'analisi matematica [...]

Torricelli, dunque, fu uno dei grandi precursori della grande svolta che portò alla nascita dell'analisi infinitesimale; egli, tuttavia, al pari di Fermat, non disponeva ancora dello strumento tecnico per formalizzare le proprie geniali intuizioni [Guido Castelnuovo, 1938]».

Può essere interessante analizzare alcuni aspetti di questo solido sia seguendo l'approccio di Torricelli nella sua Opera Geometrica sia seguendo un approccio moderno con gli strumenti disponibili dell'analisi.

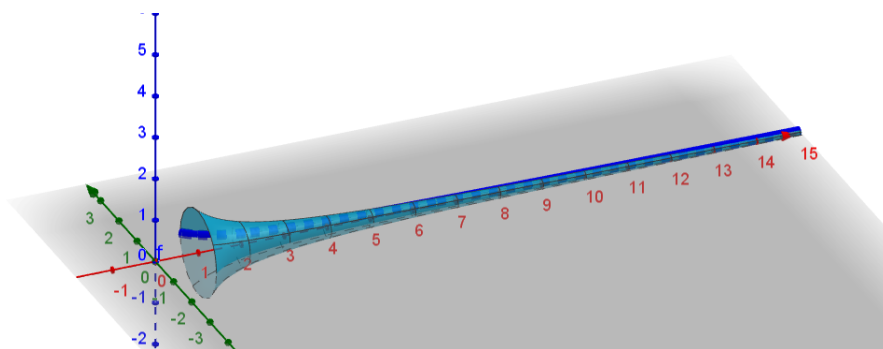
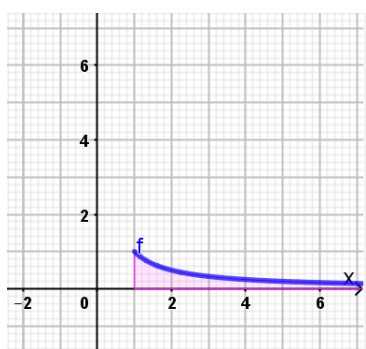
Analisi di alcune proprietà del solido: volume e superficie, usando gli strumenti moderni dell'analisi

Si può calcolare il **volume** dello spazio occupato dal solido attraverso il calcolo integrale in particolare con il metodo dei dischi o delle rondelle, Disk Method; per calcolare il volume di un solido ottenuto dalla rotazione di una regione R, delimitata dall'asse x e dal ramo di una curva di equazione $y = f(x)$ continua nell'intervallo $[a; b]$, attorno all'asse delle x si usa la formula $\int_a^b \pi \cdot f^2(x) dx$.

Nel caso del solido acuto iperbolico in cui la funzione è $f(x) = \frac{1}{x}$ nell'intervallo $[1; +\infty[$ si ottiene:

$$\text{Volume} = \int_1^{+\infty} \pi \cdot \frac{1}{x^2} dx = \lim_{M \rightarrow +\infty} \int_1^M \pi \cdot \frac{1}{x^2} dx = \lim_{M \rightarrow +\infty} \left[-\pi \cdot \frac{1}{x} \right]_{x=1}^{x=M} = \lim_{M \rightarrow +\infty} \left(-\pi \cdot \frac{1}{M} + \pi \cdot 1 \right) = \pi$$

Il seguente calcolo ha portato a dimostrare non solo che il volume del solido è finito ma anche che è uguale alla celebre costante matematica π che ha un ruolo strategico nella matematica oltre ad avere una giornata ad esso dedicata; il 14 marzo, infatti, è il "PiDay" appunto il 3/14 nell'accezione anglosassone (tale scrittura porta proprio alle prime tre cifre note del numero).

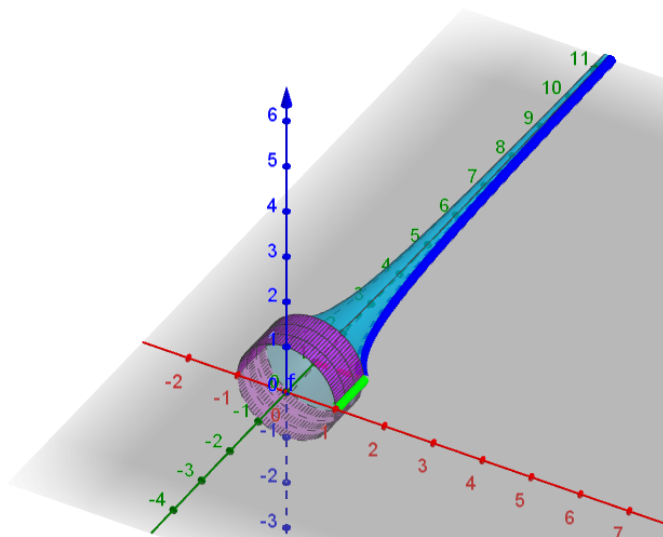


Il valore del **volume** del solido si può ottenere anche con il metodo dei gusci cilindrici, Shell Method, in cui per calcolare il volume di un solido ottenuto dalla rotazione di una regione S, delimitata dall'asse x e dal ramo di una curva di equazione $y = f(x)$ continua nell'intervallo $[a; b]$, attorno all'asse delle y si usa la formula $\int_a^b 2\pi x \cdot f(x) dx$.

Il ramo della curva $f(x) = \frac{1}{x}$ nell'intervallo $[1; +\infty[$ risulta congruente al ramo della stessa curva nell'intervallo $]0; 1]$, infatti i due rami risultano simmetrici rispetto alla bisettrice del primo e terzo quadrante. Anche il solido ottenuto ruotando il ramo della curva $f(x) = \frac{1}{x}$ nell'intervallo $[1; +\infty[$ attorno all'asse x è congruente al solido ottenuto ruotando il ramo della curva $f(x) = \frac{1}{x}$ nell'intervallo $]0; 1]$ attorno all'asse y.

Quindi se si applica tale metodo nel caso del solido acuto iperbolico in cui la funzione è $f(x) = \frac{1}{x}$ nell'intervallo $]0; 1]$ si ottiene:

$$\int_0^1 2\pi x \cdot \frac{1}{x} dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 2\pi dx = \lim_{\varepsilon \rightarrow 0^+} [2\pi x]_{x=\varepsilon}^{x=1} = \lim_{\varepsilon \rightarrow 0^+} (2\pi - 2\pi\varepsilon) = 2\pi$$



A tale valore occorre però sottrarre il volume del cilindro di base che ha raggio e altezza pari ad 1.

$$\text{Volume} = \int_0^1 2\pi x \cdot \frac{1}{x} dx - \pi = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 2\pi dx - \pi = \lim_{\varepsilon \rightarrow 0^+} [2\pi x]_{x=\varepsilon}^{x=1} - \pi = \lim_{\varepsilon \rightarrow 0^+} (2\pi - 2\pi\varepsilon) - \pi = \pi$$

In entrambi i casi si ottiene per il volume del solido acuto iperbolico un valore pari al famoso numero π .

Per il calcolo dell'estensione della **superficie laterale U del solido** ottenuto dalla rotazione di un ramo di una curva di equazione $y = f(x)$ nell'intervallo $[a; b]$, attorno all'asse delle x si usa invece la formula:

$$\int_a^b 2\pi \cdot f(x) \cdot \sqrt{1 + (f'(x))^2} dx.$$

Nel caso del solido acuto iperbolico in cui la funzione è $f(x) = \frac{1}{x}$ nell'intervallo $[1; +\infty[$ si ottiene:

$$\begin{aligned} \text{Area di U} &= \int_1^{+\infty} 2\pi \cdot \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx = \lim_{M \rightarrow +\infty} \int_1^M 2\pi \cdot \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx > \lim_{M \rightarrow +\infty} \int_1^M 2\pi \cdot \frac{1}{x} dx = \\ &= \lim_{M \rightarrow +\infty} (2\pi \cdot \ln(M) - 2\pi \cdot \ln(1)) = +\infty \end{aligned}$$

Analisi di alcune proprietà del solido usando gli strumenti di Torricelli, gli indivisibili

Nell'*Opera Geometrica* di Torricelli è presente la motivazione del volume finito del solido acuto iperbolico svolta in diverse modalità; è interessante quella che utilizza gli indivisibili di Cavalieri, in particolare risulta molto innovativo l'impiego degli indivisibili curvi nello spazio. L'altra modalità di dimostrazione proposta da Torricelli è attraverso il metodo di esaustione ma è una modalità che risulta più complessa e macchinosa, senza un approccio costruttivo ma con un approccio vicino al metodo di dimostrazione per assurdo.

Per poter comprendere il percorso proposto da Torricelli tramite gli indivisibili occorre capire cosa sono gli indivisibili e in particolare la loro trasposizione nel principio di Cavalieri.

Il famoso Principio di Cavalieri è uno degli assiomi della geometria euclidea dello spazio ed afferma che se due solidi possono essere collocati nello spazio in modo che le sezioni dei due solidi con i piani paralleli ad un piano dato sono equivalenti (cioè hanno stessa area) allora i due solidi sono equivalenti (cioè hanno lo stesso volume).

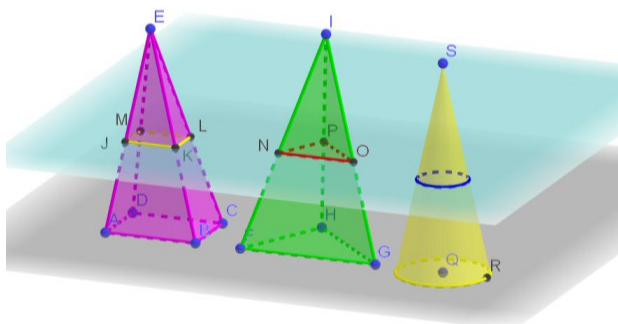


Figura 1 Solido 1 Solido 2 Solido 3

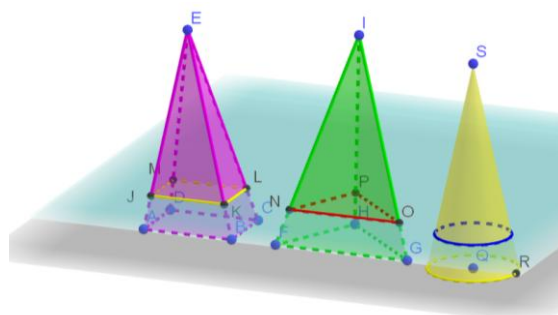


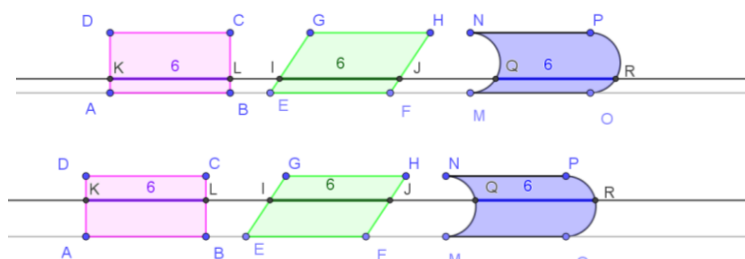
Figura 2 Solido 1 Solido 2 Solido 3

Nelle figure soprastanti i tre solidi hanno sezioni di uguale estensione con un qualsiasi piano parallelo al piano di base quindi i tre solidi hanno anche stesso volume (nelle figure sono rappresentati due diversi piani: le tre sezioni della figura 1 sono equivalenti fra loro, le tre sezioni della figura 2 sono equivalenti fra loro ma non sono equivalenti a quelle della figura 1).



Nella figura soprastante sono state realizzate pile/solidi di 12 monete da 20 centesimi; sono esempi per mostrare una realizzazione concreta del principio di Cavalieri nello spazio: tutti i solidi, avendo sezioni congruenti con i piani paralleli al piano del pavimento, hanno lo stesso volume pur avendo forme molto diverse.

Esiste anche la versione nel piano del Principio di Cavalieri che afferma che se due figure possono essere collocate nel piano in modo che le intersezioni delle due figure con le rette parallele ad una retta data sono congruenti (cioè hanno la stessa lunghezza) allora le due figure sono equivalenti (cioè hanno la stessa area).



Nelle immagini soprastanti si vuole evidenziare che ogni intersezione tra le figure e le rette parallele ad una retta voluta hanno la stessa lunghezza e quindi le tre figure hanno tutte la stessa area.

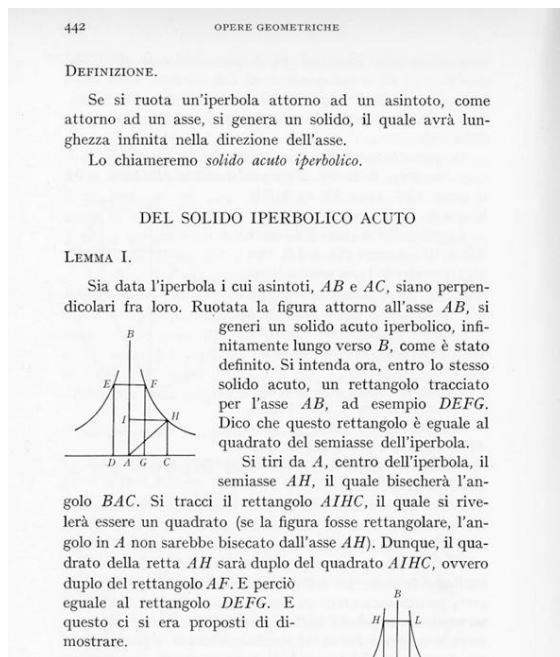


Nella figura soprastante sono state realizzate figure/file con 14 pennarelli; sono esempi per mostrare una realizzazione concreta del principio di Cavalieri nel piano: tutte le figure, avendo sezioni congruenti con rette parallele alla direzione fissata, hanno la stessa area pur avendo forme molto diverse.

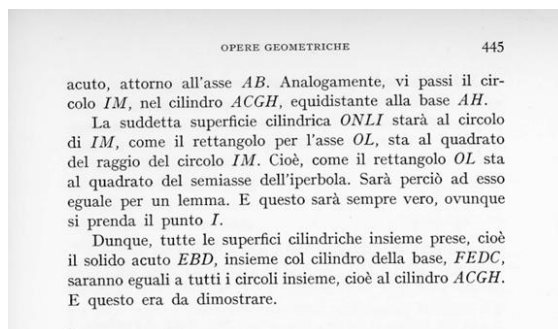
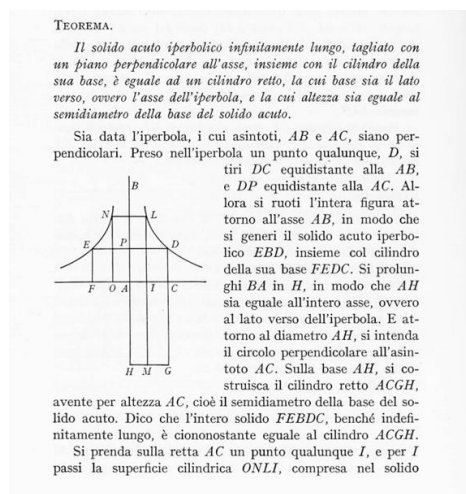
Come accennato in precedenza l'importanza del contributo di Torricelli allo studio di questo solido risiede anche nell'applicazione di indivisibili curvi nello spazio.



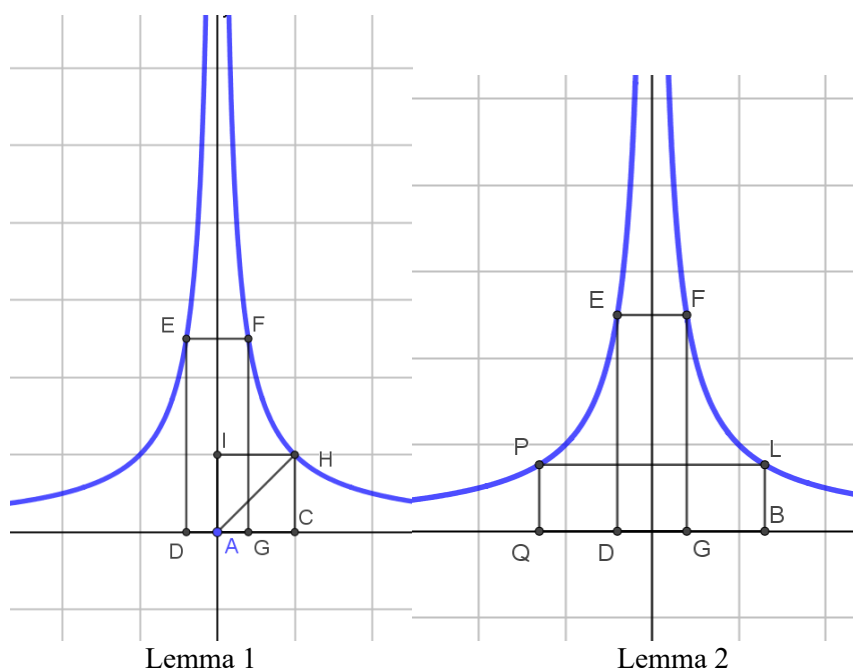
Opera Geometrica originale di Torricelli



Traduzione dell'Opera Geometrica di Belloni



In termini moderni si deve considerare il ramo della curva $f(x) = \frac{1}{x}$ nell'intervallo $]0; 1]$ mentre nelle immagini sottostanti come nell'*Opera Geometrica* viene considerato un ramo di maggiore estensione per mostrare meglio il ragionamento che viene seguito.



Si considera il solido acuto iperbolico ottenuto ruotando il ramo di iperbole attorno all'asse y . Si ripercorre in sintesi l'idea della dimostrazione di Torricelli senza entrare in tutti i dettagli e ci si muove tra l'approccio originale e la visione moderna della geometria analitica piana e solida. Nel Lemma 1 dell'*Opera Geometrica* Torricelli osserva come i rettangoli inscritti nella curva, formata dal ramo di iperbole e dalla sua simmetrica rispetto all'asse y , come per esempio il rettangolo di base DG e altezza GF (in termini moderni tale rettangolo ha base di lunghezza pari a $2t$ e altezza di lunghezza pari a $1/t$, quindi l'area del rettangolo risulta pari a 2) sono equivalenti al quadrato avente per diagonale il segmento AH , che è pari al semiasse dell'iperbole (in termini moderni il semiasse dell'iperbole $xy = 1$, essendo $\frac{a^2}{2} = 1$, risulta essere uguale ad $a = \sqrt{2}$, quindi l'area del quadrato è pari a 2).

Nel Lemma 2 Torricelli osserva che siccome tutti i rettangoli precedenti hanno la stessa area allora se messi in rotazione attorno all'asse y generano cilindri inscritti nel solido acuto iperbolico che hanno tutti la stessa superficie laterale (in termini moderni tali cilindri hanno tutti superficie laterale: $2\pi \cdot R \cdot h = 2\pi \cdot t \cdot \frac{1}{t} = 2\pi$).

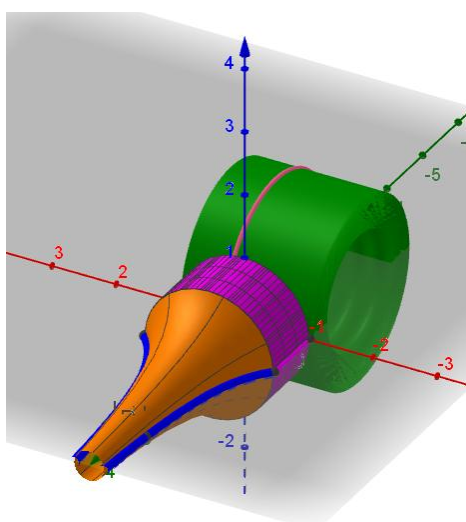
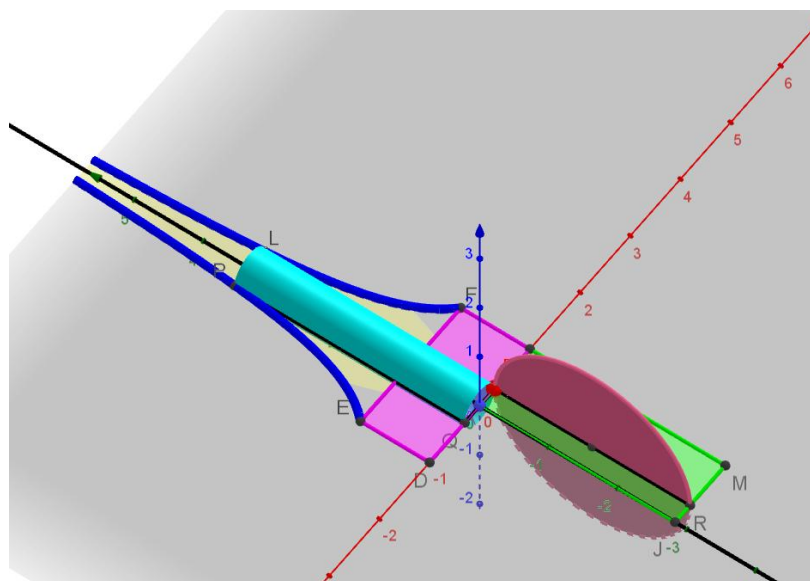
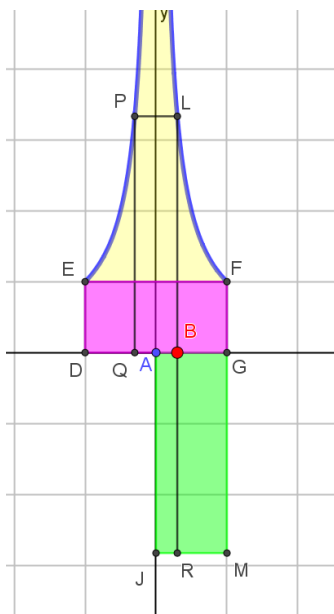
Nei successivi lemmi si segue il ragionamento che porta al teorema cruciale che enuncia (trasposizione di Belloni): « Il solido acuto iperbolico infinitamente lungo, tagliato da un piano perpendicolare all'asse, insieme al cilindro della sua base, è uguale al cilindro retto la cui base sia il lato verso, ovvero l'asse dell'iperbole, e la cui altezza sia uguale al semidiametro della base del solido acuto. »

Tradotto nel linguaggio moderno nel teorema si afferma che il solido $S1$ formato dal solido acuto iperbolico e dal cilindro "di base" (su cui si appoggia il solido acuto iperbolico) ha lo stesso volume di un altro cilindro retto $S2$ che ha diametro di base uguale all'asse dell'iperbole e ha altezza uguale al semidiametro della base del solido acuto iperbolico.

Per dimostrare questa equivalenza Torricelli si avvale degli indivisibili curvi e mostra come gli indivisibili che formano il solido $S1$ (formato dal solido acuto iperbolico insieme al cilindro di base) hanno la stessa area degli indivisibili curvi che formano il solido $S2$.

Gli indivisibili curvi tridimensionali con cui Torricelli costituisce il solido $S1$ sono superfici cilindriche cave con raggio variabile AB e altezza variabile BL (in termini moderni il raggio AB risulta pari a t e l'altezza BL risulta pari ad $1/t$ quindi la superficie laterale di queste superfici cilindriche è data da $2\pi \cdot R \cdot h = 2\pi \cdot t \cdot \frac{1}{t} = 2\pi$).

Gli indivisibili curvi bidimensionali con cui Torricelli costituisce il solido $S2$ sono cerchi con raggio pari al semiasse dell'iperbole (in termini moderni il diametro BR di tali cerchi è sempre uguale all'asse dell'iperbole, cioè è uguale a $2\sqrt{2}$, quindi l'area di tali cerchi è data da $\pi r^2 = \pi(\sqrt{2})^2 = 2\pi$).



È molto interessante il confronto tra gli indivisibili cilindrici tridimensionali di S1 con indivisibili circolari piani di S2, al variare del punto B (evidenziato nella figura precedente): le dimensioni degli indivisibili cilindrici variano sia la posizione che la forma (al crescere del raggio diminuisce l'altezza) mentre gli indivisibili circolari cambiano posizione ma rimangono sempre congruenti fra loro.

Grazie all'uguale superficie di tutti i possibili indivisibili dei solidi S1 e S2 si conclude che i volumi dei due solidi sono uguali. E' quindi possibile trovare il volume V del solido acuto iperbolico per differenza tra il volume del solido S2 e il volume del cilindro di base a cui si appoggia il solido acuto iperbolico.

$$V = 2\pi - \pi = \pi$$

È molto affascinante notare come l'idea innovativa di Torricelli degli indivisibili cilindrici tridimensionali sia ancora presente nell'attuale metodo dei gusci cilindrici esaminato in precedenza per il calcolo dei volumi di particolari solidi di rotazione.

Gli indivisibili e le stampanti 3D

Le attuali stampanti 3D costruiscono gli oggetti sovrapponendo strati di opportuni materiali a partire da un piano di base, muovendosi dal basso verso l'alto seguendo le istruzioni contenute in opportuni file. Il materiale con cui vengono realizzati gli strati può essere di diversa natura (filamento, resina liquida, polvere, ecc.).

L'oggetto da stampare viene "affettato" da opportuni software in tanti strati molto sottili che verranno poi riprodotti dalla stampante 3D seguendo le istruzioni del software; tali strati costituiscono una buona

approssimazione e concretizzazione degli indivisibili di cui si è parlato in precedenza, gli strati sono quindi le sezioni piane dell'oggetto con vari piani paralleli al piano di base.

E' possibile realizzare copie fisiche del solido acuto iperbolico proprio utilizzando una stampante 3D; le istruzioni necessarie si possono facilmente reperire in rete. Si possono vedere alcuni modelli realizzati dal docente Michele Castellari del Liceo Torricelli nelle figure seguenti, e anche altri oggetti realizzati, su commissione, dalla Palestra della Scienza di Faenza.



Il solido acuto iperbolico a volte viene anche chiamato “solido iperbolico acutissimo” o “anfora di Zeus” o “Tromba di Torricelli” o “Tromba di Gabriele”, l'enciclopedia on-line riporta la motivazione del riferimento all'arcangelo Gabriele: “l'angelo che, secondo tradizione, soffiò nel corno per annunciare l'apocalisse, associando il divino (e quindi l'infinito, come l'estensione della superficie del solido) all'umano (quindi al finito, come il volume dello spazio occupato dal solido)”.

Nella storia ci sono stati altri oggetti stupefacenti con caratteristiche in apparenza contraddittorie, oltre alla Tromba di Torricelli. Un celebre esempio è il fiocco di neve Koch, una meravigliosa figura che racchiude un'area finita ma che ha contorno di lunghezza infinita. Un altro esempio, sempre proveniente dal mondo dei frattali, è la spugna di Menger un oggetto con volume nullo e con facce che hanno area nulla ma perimetro infinito.

La matematica è un linguaggio ed uno strumento che permette di andare oltre all'intuizione e di esplorare nuovi orizzonti e raggiungere sempre nuovi traguardi; la stessa idea che può nascere osservando alcune opere del celebre maestro Maurits Cornelis Escher (come Circle Limit IV, Serpenti, Vortici, e molte altre).



Il gatto Eulero e la tromba di Torricelli: la matematica affascina la curiosità.

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Ritorna all'indice

Un esperimento dimostrativo della legge di Torricelli

Stefano Alberghi
Società Torricelliana di Scienze e Lettere
Liceo Torricelli, Faenza

Il cosiddetto teorema di Torricelli o “legge di Torricelli” descrive il comportamento di un liquido che fuoriesce da un foro posto sulla superficie laterale di un recipiente. In genere si intende che il foro sia posizionato in prossimità della base del recipiente, e sia di dimensioni tali per cui il liquido che esce non alteri sensibilmente la quantità, e quindi l'altezza, del liquido contenuto in esso. Il teorema può essere espresso in questi termini: la velocità di fuoriuscita del liquido è uguale alla velocità che un corpo, caduto da un'altezza pari alla superficie superiore del liquido, avrebbe quando si trova all'altezza del foro; detto in formule: la velocità v di un liquido che esce da un piccolo foro laterale in un recipiente è pari a $\sqrt{2gh}$, dove h è la distanza tra la superficie libera del liquido e il foro stesso.

Torricelli nella sua *Opera Geometrica*, pubblicata nel 1644 (in particolare nel trattato *De motu aquarum*) lo esprime così:

“Supponiamo che l'acqua, che erompe con violenza da un orifizio, abbia, nel punto della sua fuoriuscita, lo stesso impeto, che avrebbe un grave, ovvero una goccia della stessa acqua, che fosse caduto dalla superficie superiore del liquido, fino all'orifizio.”

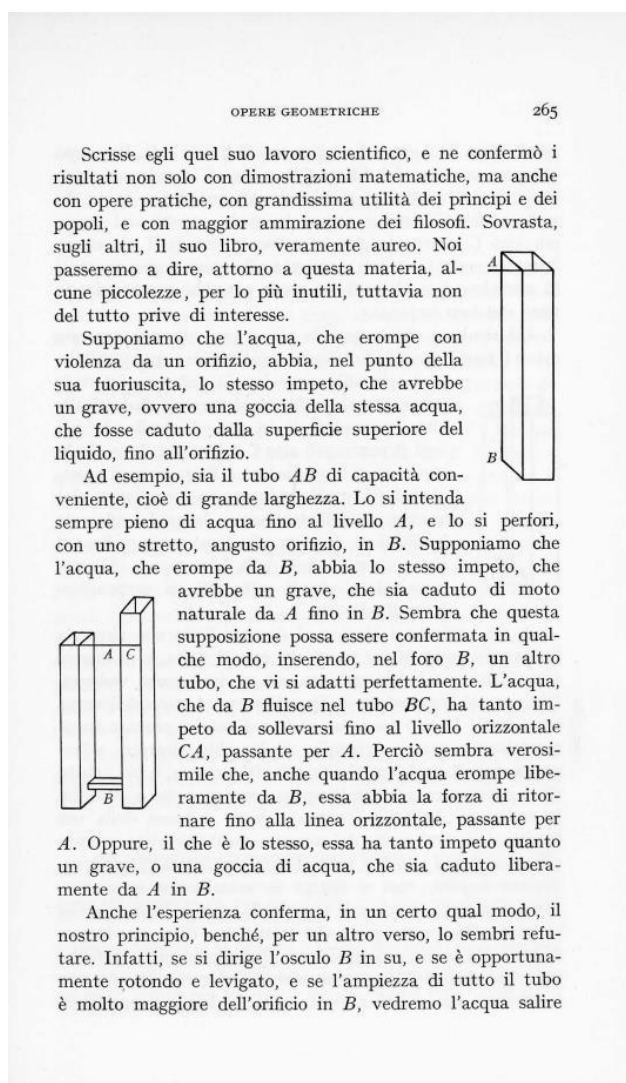
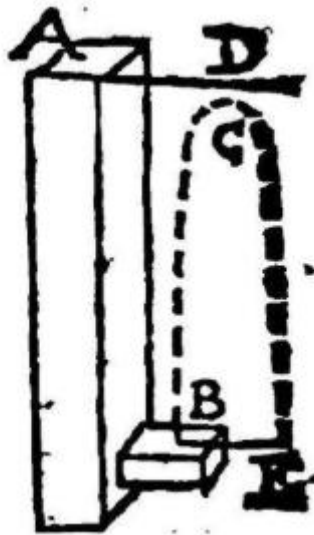


Figura da “Opere scelte di Evangelista Torricelli”, a cura di Lanfranco Belloni, UTET 1975

Più avanti nello stesso testo, Torricelli, per fornire un'evidenza osservativa e misurabile di quanto enunciato, ipotizza di collegare al foro un piccolo tubo rivolto all'insù e di notare come l'acqua dello zampillo che ne fuoriesce giunga all'altezza della superficie superiore del liquido. In ciò si sta riferendo alla teoria dei gravi di Galileo per cui, in assenza di attrito, un corpo che cade da un'altezza h lungo un qualsiasi piano inclinato acquisterà una velocità tale da poter risalire lungo lo stesso piano fino all'altezza da cui era partito. Evidentemente nella pratica ciò non avviene per via dell'attrito, sia per quanto riguarda i solidi sia per quanto riguarda i liquidi. Torricelli stesso osserva che, eseguendo l'esperimento, l'acqua non arriva all'altezza sperata, ma che si riscontra un "difetto", imputabile agli attriti con l'aria o con l'acqua stessa.



Ad ogni modo nella fisica odierna il “teorema di Torricelli” è considerato corretto nella sua formulazione, in quanto tra l'altro è stato dimostrato essere un caso particolare di una legge ben più generale sul moto dei fluidi, ovvero il teorema di Bernoulli, illustrato dal matematico svizzero Daniel Bernoulli nella sua opera *Hydrodynamica* del 1738. Tale teorema a sua volta consiste nell'applicazione ai fluidi del noto (e ancora più generale) principio di conservazione dell'energia meccanica, per cui è costante la somma delle energie cinetica (dovuta alla velocità di un corpo) e potenziale (dovuta ad esempio dall'altezza a cui il corpo si trova).

Per i solidi puntiformi (ad esempio una piccola biglia) tale principio si scrive in questo modo:

$$(1) \quad \frac{1}{2}mv^2 + mgh = \text{costante}$$

Il primo termine è l'energia cinetica del solido (m è la massa, v la velocità), il secondo è l'energia potenziale gravitazionale (g è legato alla gravità, h è l'altezza a cui si trova il solido). La loro somma in assenza di attriti è costante. Questo fatto è vero indipendentemente dal percorso della biglia, sia esso verticale o obliquo, rettilineo o curvo (v. figura sotto). L'energia totale è sempre la stessa in tutti i punti del percorso, e quindi anche in quelli iniziale A e finale B per cui

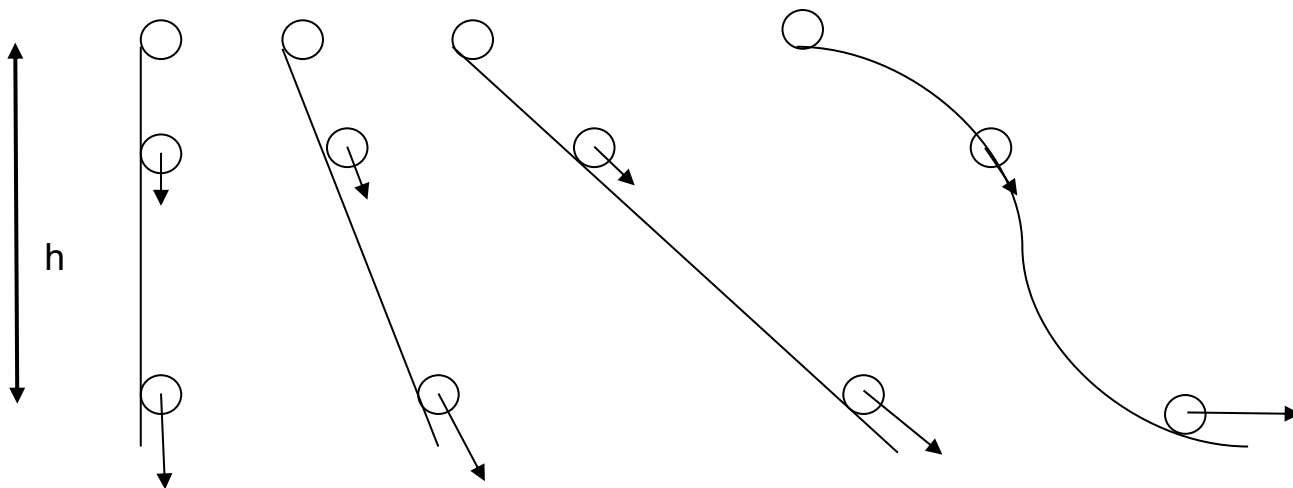
$$(1\text{bis}) \quad \frac{1}{2}mv_A^2 + mgh_A = \frac{1}{2}mv_B^2 + mgh_B$$

Da questa formula, nel caso in cui la biglia venga fatta scivolare partendo da ferma (e quindi $v_A=0$) si ricava con pochi passaggi la velocità finale della biglia quando arriva a terra ($h_B=0$)

$$(1\text{ter}) \quad mgh_A = \frac{1}{2}mv_B^2$$

e dunque

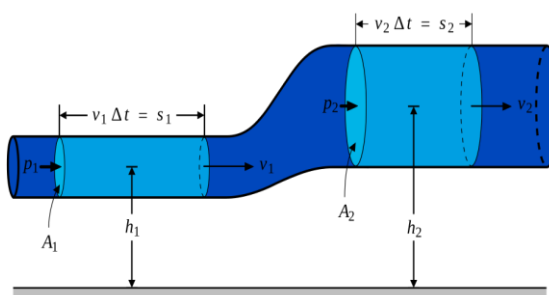
$$(2) \quad v_B = \sqrt{2gh_A}$$



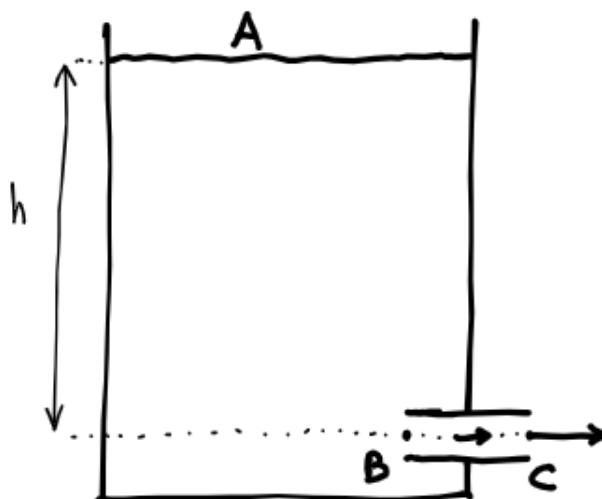
Per i fluidi, ad esempio liquidi in moto dentro una tubatura, oltre alla velocità e all'altezza del fluido, può variare anche la forma del contenitore, ad esempio la sezione del tubo, e la pressione del fluido nei vari punti del suo percorso, per cui l'equazione è leggermente più complicata. La sua formulazione è la seguente:

$$(3) \quad P + \frac{1}{2}dv^2 + dgh = \text{costante}$$

dove il primo termine rappresenta la pressione, il secondo l'energia cinetica (d è la densità, mentre v è sempre la velocità), il terzo l'energia potenziale gravitazionale (h è l'altezza del fluido in un dato punto del suo percorso). La somma di questi tre termini, in assenza di attrito (quindi per fluidi non viscosi), è costante.



Per ricavare la legge di Torricelli da quella di Bernoulli si può considerare un contenitore pieno di acqua come quello in figura sotto, dove il foro viene rappresentato come un piccolo tratto di tubo.



In tale contenitore l'acqua è sostanzialmente ferma per cui la velocità v è nulla ovunque ($v = 0$) e in particolare nel punto B all'ingresso del foro. All'uscita del foro l'acqua assume una velocità v_C , che può essere calcolata dall'equazione (3).

Confrontiamo ed eguagliamo i termini di tale equazione, calcolati separatamente nei punti A e C

$$(4) \quad P_A + \frac{1}{2} \rho v_A^2 + \rho g h_A = P_C + \frac{1}{2} \rho v_C^2 + \rho g h_C$$

La pressione P del fluido nei punti in cui esso è a contatto con l'aria è esattamente uguale a quella dell'aria, cioè alla pressione atmosferica. Per tale motivo nell'equazione il termine P relativo alla pressione in A è identico a quello in C e possiamo non considerarlo, limitandoci a confrontare gli altri

$$(5) \quad \frac{1}{2} \rho v_A^2 + \rho g h_A = \frac{1}{2} \rho v_C^2 + \rho g h_C$$

Il termine relativo alla velocità (energia cinetica) non è presente in A ma solo in C, mentre quello relativo all'altezza (energia potenziale) non è presente in C ma solo in A. Basterà dunque che siano uguali questi due termini e cioè:

$$(6) \quad \rho g h_A = \frac{1}{2} \rho v_C^2$$

La densità può essere eliminata, essendo presente in entrambi i termini, e dunque otteniamo

$$(7) \quad g h_A = \frac{1}{2} v_C^2$$

e da questa nuovamente la relazione (2) $v_C = \sqrt{2gh_A}$ ottenuta per la biglia.

Dunque, le velocità della biglia e dell'acqua uscente dal foro sono uguali e in questo modo viene dimostrato il teorema di Torricelli.

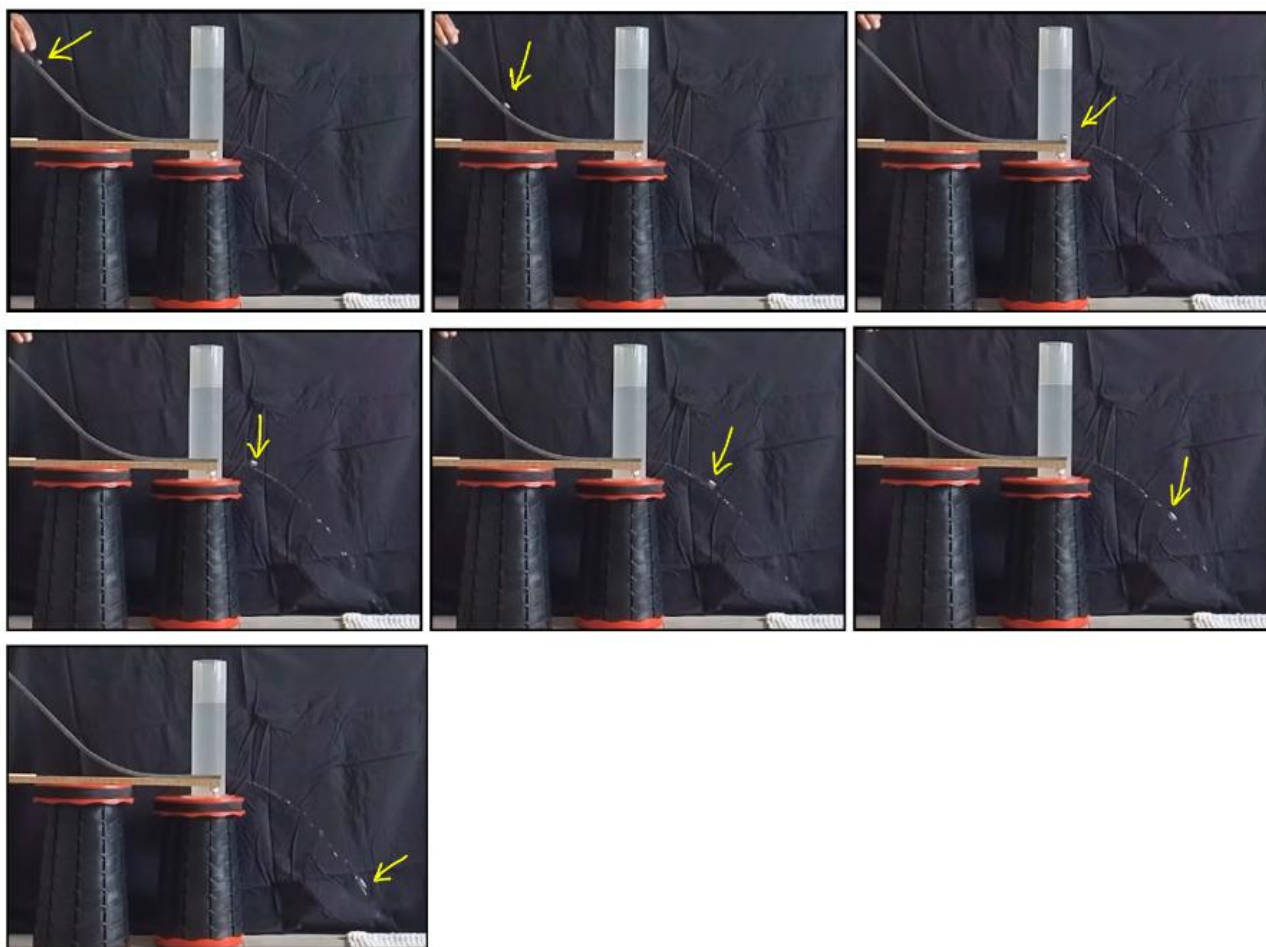
Abbiamo voluto verificarlo tramite un piccolo esperimento qualitativo, diverso però da quello pensato da Torricelli. Nel nostro caso vogliamo confrontare direttamente, tramite fotocamera, la velocità dello zampillo d'acqua che fuoriesce dal contenitore e la velocità di una biglia lasciata rotolare su una rampa: la biglia viene fatta partire da ferma allo stesso livello della superficie libera dell'acqua e fatta uscire dalla rampa orizzontalmente allo stesso livello del foro. In tal modo, se il teorema è verificato, sia la biglia sia lo zampillo vengono lanciate entrambe orizzontalmente con la medesima velocità. Allineando i punti di lancio possiamo confrontare le rispettive traiettorie paraboliche che, come noto, e in assenza di attrito, non dipendono dalla massa del corpo ma solo dalla velocità e direzione di partenza. Il risultato è stato soddisfacente, come si può vedere dai fotogrammi sotto riportati che mostrano due traiettorie paraboliche sostanzialmente coincidenti in ogni punto.

Si fa presente che ci sono ulteriori variabili da considerare che rendono l'esperimento semplicemente "dimostrativo": ad esempio gli attriti con l'aria, l'attrito della biglia con la pista, la viscosità dell'acqua. Inoltre, la biglia ha una energia di rotazione, pur trascurabile, che sottrae inevitabilmente energia al moto traslatorio.

L'immagine è stata ripresa da vicino per cui vi sono alcuni problemi di parallasse. L'altezza di partenza della biglia, pur essendo uguale a quella del livello dell'acqua, sembra leggermente superiore: ciò è dovuto al fatto che la rampa per la biglia è più vicina all'obiettivo della fotocamera, la quale è posta più in basso (la fotocamera è infatti posizionata circa al livello del foro).

Il video dell'esperimento è disponibile al seguente link:

<https://photos.app.goo.gl/jS589fS9ge8THZ1r8>



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